ORLICZ SPACES

DISSERTATION

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

IN

MATHEMATICS

By

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2004

Dedicated

To my

Parents

DEPARTMENT OF MATHEMATICS

ALIGARH MUSLIM UNIVERSITY, ALIGARH-202002 (INDIA)

Dated: 18.11.2004

Prof. Mursaleen

Certificate

This is to certify that the dissertation entitled ***Orlicz Spaces*** has been carreid out by ***Mr. Syed Abdul Mohiuddine*** under my supervision and work is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.

It is further certified that work has not been submitted in any university or institution for the award of any other degree or diploma

Mursaleen

Prof. Mursaleen

(Supervisor)

Acknowledgement

*It has been my proud privilege to have accomplished my dissertation under the able guidance of **Prof. Mursaleen,** Department of Mathematics, Aligarh Muslim University, Aligarh. I have a great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and high indebtedness to **Prof. Mursaleen,** to whom I owe more than I can possibly express, for his inspiring supervision, constant help and encouragement throughout to complete this dissertation.*

*I am greatly indebted to **Prof. M.Z. Khan,** Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, for providing necessary facilities and help during my M. Phil work.*

*I am also very much grateful to all of my friends especially **Mr. Mohd. Haroon Khan** and colleagues especially **Mr. Vakeel Ahmad Khan** who always appeared helpful to me, whenever I needed.*

*I am further grateful to **Mr. Abul Irfan Azad** for taking pains to finish the typing of my dissertation in time.*

Last but not the least, my special thanks and gratitude must go to my parents, brothers and sisters for their inspiration and continuous encouragement throughout my life. I express my infinite indebtedness to them, whose love, financial and moral supports are the base of my every success.

(Syed Abdul Mohiuddine)

PREFACE

The present dissertation entitled **"ORLICZ SPACES"** has been written for submission to Aligarh Muslim University in partial fulfilment of the requirements of the degree of Master of Philosophy.

The dissertation consists of six chapters. In chapter I, we recall notations, convexity, uniform convexity, modular space, Orlicz function, Musielak-Orlicz function, δ_2-condition, Orlicz spaces, Orlicz and Luxemburg norm, paranorm and seminorm.

In chapter II, we discuss the criteria for rotundity, local uniform rotundity, compact local uniform rotundity and property H in Orlicz sequence spaces equipped with the Orlicz norm.

In chapter III, for a Banach space X, the Orlicz vector-valued sequence space $l_M(X)$ and its modular are defined and it is proved that $l_M(X)$ is rotund if and only if X is rotund, $M \in \delta_2$ and M is strictly convex on $[0, M^{-1}(1/2)]$.

In chapter IV, we study the criteria for the property (β), local uniform convexity, local weak uniform convexity and local property (β) in Musielak-Orlicz sequence spaces.

In chapter V, we define the strong summability by a Musielak-Orlicz function and examine its relationship with A-statistical convergence.

Finally in the last chapter we introduce some new sequence spaces with respect to an Orlicz function.

Towards the end of the dissertation, we have given a fairly exhaustive bibliography of the books and publication to which reference have been made throughout the dissertation.

CONTENTS

CHAPTER-I

DEFINITIONS AND NOTATIONS

The primary aim of this section is to recall some basic notations and definitions of which we shall make frequent use in the rest of this work without further references

1.1. Notations

The symbols are used to denote:

$\mathbb{N}$: Set of all positive integers

$\mathbb{R}$: Set of all real numbers

$\mathbb{C}$: Set of all complex numbers

A : The matrix $(a_{nk})_{n,k=1}^{\infty}$

$\|A\|$: $\sup_n \sum_k |a_{nk}|$

$x = (x_k)$: any sequence whose k^{th} term is x_k.

$p = (p_k)$: a sequence of strictly positive real numbers with

$$\sup_k p_k < \infty$$

$e_k = (0,0,0,\ldots,1$ (k^{th} place)$,0,0,0,..)$ for all $k \in \mathbb{N}$

$e = (1,1,1,\ldots\ldots,)$

l^0 or $\omega := \{x = (x_k)\colon x_k \in \mathbb{R}$ or $\mathbb{C}\}$, the space of all real or complex sequences

l_∞:= $\{x = (x_k)\colon \sup_k |x_k| < \infty\}$, the space of all bounded sequences,

normed by $\|x\|_\infty = \sup_k |x_k|$

c:= $\{x = (x_k)\colon \lim_k x_k = l,\ l \in \mathbb{C}\}$, the space of all convergent

sequences, normed by $\|x\|_\infty = \sup_k |x_k|$

c_0:= $\{x = (x_k)\colon \lim_k x_k = 0\}$, the space of all null sequences,

normed by $\|x\| = \max_k |x_k|$ or $\sup_k |x_k|$

l_1:= $\{x = (x_k)\colon \sum_k |x_k| < \infty\}$, the space of absolutely convergent

series, normed by $\|x\| = \sum_k |x_k|$

l_p:= $\{x = (x_k)\colon \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ $(p > 0)$, the space of absolutely

p-summable sequences, normed by $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$, $\quad 1 \le p < \infty$

if $0 < p < 1$, $\|x\|_p = \sum_k |x_k|^p$ $\quad$ (p-norm)

L_p:= The Lebesgue integration space $L_p\,[a,b]$, $1 \le p < \infty$ is the Banach space whose elements are real-valued functions x on $[a,b]$ such that

$$\int_a^b |x(t)|^p \, dt < \infty$$

where the integral is taken in the Lebesgue sense.

The Banach space $L_p\,[a,b]$ is the completion of the normed space which consists of all continuous real-valued functions on $[a,b]$ and the norm defined by

$$\|x\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{1/p}$$

The elements of $L_p\,[a,b]$ are equivalence classes of those functions, where x is equivalent to y if the Lebesgue integral of $|x-y|^p$ over $[a,b]$ is zero.

1.2. Convexity

A continuous function $M\colon \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$M\left(\frac{u+v}{2} \right) \le \frac{M(u)+M(v)}{2} \qquad \text{for all } u,v \in \mathbb{R}.$$

If in addition, the two sides of above are not equal for $u \neq v$ then we call M *strictly convex.*

1.3. Uniform Convexity

A continuous function $M\colon \mathbb{R} \to \mathbb{R}$ is said to be *uniformly convex* if for any $\varepsilon > 0$ and any $u_0 > 0$ there exists some $\delta > 0$ such that

$$M\left(\frac{u+v}{2} \right) \le (1-\delta) \frac{M(u)+M(v)}{2}$$

for all $u,v \in \mathbb{R}$ satisfying $|u-v| \geq \varepsilon \max\{|u|,|v|\} \geq \varepsilon u_0$

1.4. Modular Space

A function $\rho : X \to [o,\infty]$ is called a *modular* if it satisfies the following conditions:

1. $\rho(x) = 0$ if and only if $x = 0$
2. $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x,y \in X$ and all $\alpha,\beta > 0$ with $\alpha+\beta = 1$.

The modular ρ is called *convex* if

4. $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\,\rho(y)$, for all $x,y \in X$ and all $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$

For any modular ρ on X, the space $X_\rho = \{x \in X : \rho(\lambda x) < \infty$ for some $\lambda > 0\}$ is called the *modular space.*

1.5. Orlicz Function

A map $M:\mathbb{R} \to [0,\infty]$ is said to be an *Orlicz function* if it is even, convex, continuous and vanishing at 0 and $M(u) \to \infty$ as $u \to \infty$.

We say that an Orlicz function M is an N'- *function* if it satisfies the following condition:

$$\lim_{u \to \infty} \frac{M(u)}{u} = \infty$$

An N'-function M is said to be an *N-function* if

$$\lim_{u \to 0} \frac{M(u)}{u} = 0 \qquad (1.5.1)$$

If the convexity of the Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called *modulus function* introduced by Ruckle [41] and studied by Maddox [33] and others.

1.6. Musielak-Orlicz Function

A sequence $\mathcal{M}=(M_i)$ of Orlicz functions is called a *Musielak-Orlicz function.* By $\mathcal{N}=(N_i)$ we denote the complimentary function of $\mathcal{M}$ is sense of Young, i.e.,

$N_i(v) = \sup\{|v|u - M_i(u) : u \geq 0\},\ i = 1,2,....$

For given Musielak-Orlicz function $\mathcal{M}$, we define a *convex modular* by $I_{\mathcal{M}}(x) = \sum_{i=1}^{\infty} M_i(x(i))$ for any $x = (x\ (i)) \in l^0$.

We define *Musielak- Orlicz sequence space* $l_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ by

$l_{\mathcal{M}}=\{x \in l^0 : I_{\mathcal{M}}(kx) < \infty$, for some k > 0$\}$

$h_{\mathcal{M}}=\{x \in l_{\mathcal{M}}: I_{\mathcal{M}}(kx) < \infty$, for all $k > 0\}$.

1.7. δ_2-condition

We say that an Orlicz function M satisfies the *δ_2-condition* ($M \in \delta_2$ for short) if there exist constant $K \geq 2$ and $u_0 > 0$ such that $0 < M(u_0) < \infty$ and

$M(2u) \leq KM(u)$

whenever $|u| \leq u_0$

If M satisfies the δ_2 condition for any $a > 0$ with $K \geq 2$ dependent on a, we say that M satisfies the *strong δ_2-condition* (write $M \in \delta_2^S$ for short).

$\mathcal{M}$ satisfies *δ_2^*-condition* ($\mathcal{M} \in \delta_2^*$ for short) if and only if $\mathcal{N} \in \delta_2$.

An Orlicz function M is said to satisfy *δ_λ-condition* for all values u if there exists a constant $K > 0$ such that

$M(\lambda u) \leq K\lambda M(u)$ for all $u \geq 0$ and $\lambda > 1$

Remark 1.7.1. Note that neither δ_λ-condition implies δ_2-condition nor conversely. For example, let $M(u) = u^2$. Then it satisfies δ_2-condition with K=4, but for λ=5 it does not satisfy δ_λ-condition. On the other hand if K=3 and λ=2, then M satisfies δ_λ-condition but not δ_2-condition.

Theorem 1.7.2. The following are equivalent:

1. $M \in \delta_2$,
2. There exist $t > 1$, $u_0 > 0$ and $K > 1$ such that

   $M(tu) \leq KM(u) \qquad (|u| < u_0)$
3. For any $t_1 > 1$ and $u_1 > 0$, there exists $K' > 0$ such that the inequality in (2) holds for $t = t_1$, $u_0 = u_1$ and $K = K'$.

1.8. Orlicz Spaces

We always denote by (G,Σ,μ) the Lebesgue measure space in a Euclidean space with $0 < \mu G < \infty$ and by M,N a pair of Orlicz functions complementary to each other. Moreover, for a measurable function u on G, we introduce its *modular* by

$$\rho_M(u) = \int_G M(u(t))dt .$$

Then the *Orlicz space* L_M and its subspace E_M are defined as follows:

$$L_M = \{u : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\},$$

$$E_M = \{u : \rho_M(\lambda u) < \infty \text{ for all } \lambda > 0\}.$$

We define *Orlicz sequence space* l_M and its subspace h_M by

$$l_M = \{x \in l^0 : \rho_M(cx) = \sum_{i=1}^{\infty} M(cx(i)) < \infty \quad \text{for some } c > 0\}$$

$$h_M = \{x \in l_M : \rho_M(cx) = \sum_{i=1}^{\infty} M(cx(i)) < \infty \quad \text{for all } c > 0\}.$$

1.9. Orlicz and Luxemburg Norm

For each $u \in L_M$, let

$$\| u \|_0 = \sup\left\{\int_G u(t)v(t)dt : \rho_N(v) \leq 1\right. .$$

Then it is easily verified that $(L_M, \|\cdot\|_0)$ and $(E_M, \|\cdot\|_0)$ are Banach spaces. We call $(L_M, \|\cdot\|_0)$ the *Orlicz space generated by the Orlicz function M*, and $\|\cdot\|_0$ the *Orlicz norm*.

For an Orlicz space L_M, we call the functional

$$\| u \| = \inf\{\lambda > 0 : \rho_M(u/\lambda) \leq 1\} \qquad (u \in L_M)$$

the *Luxemburg norm.*

We will consider l_M equipped with the Luxemburg norm

$$\| x \| = \inf\left\{\varepsilon > 0 : \rho_M\left(\frac{x}{\varepsilon}\right) \leq 1\right\}$$

or with the equivalent one

$$\| x \|_0 = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx)),$$

called the *Orlicz norm* or the *Amemiya norm.*

(In short, we write $(l_M, \|\cdot\|) = l_M$ and $(l_M, \|\cdot\|_0) = l_M^0$).

1.10. Paranorm

A *paranorm* is a function $g: X \to \mathbb{R}$ defined on a linear space X such that for all $x, y \in X$

1. $g(x) = 0$ if $x = \emptyset$,
2. $g(-x) = g(x)$,
3. $g(x+y) \leq g(x) + g(y)$,
4. If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and x_n, $a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\lambda_n x_n \to \lambda a$ in the sense that $g(\lambda_n x_n - \lambda_a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called a *total paranorm* on X, and the pair (X,g) is called a *totally paranormed space*.

1.11. Seminorm

A *seminorm* is a functions $\nu : X \to \mathbb{R}$ defined on a linear space X such that for all $x,y \in X$

1. $\nu(x) = 0$ if $x = \emptyset$,
2. $\nu(\alpha x) = |\alpha| \nu(x)$, for all scalar α,
3. $\nu(x + y) \leq \nu(x) + \nu(y)$.

Remark 1.11.1. Every seminormed space is a paranormed but not converscely.

CHAPTER-II

ORLICZ SEQUENCE SPACES EQUIPPED WITH ORLICZ NORM

1. Preliminaries and Introduction

Let $(X, \|.\|)$ be a real Banach space, $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X respectively. By X^* denote the dual space of X. Clarkson [4] introduced the concept of uniform rotundity.

A Banach space X is said to be *uniformly rotund* (UR for short) if for every sequences (x_n) and (y_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \| x_n + y_n \| = 2$, there holds $\lim\limits_{n\to\infty} \| x_n - y_n \| = 0$.

A Banach space X is said to be *rotund* (R for short) if for any x and y in $S(X)$ with $\| x+y \| = 2$, we have $x = y$.

A Banach space X is called *locally uniformly rotund* (LUR for short) if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \| x_n + x \| = 2$, there holds $\lim\limits_{n\to\infty} \| x_n - x \| = 0$.

A Banach space X is said to be *compactly locally uniformly rotund* (CLUR for short) if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim\limits_{n\to\infty} \| x_n + x \| = 2$, it follows that the set $\{x_n\colon n \in \mathbb{N}\}$ is relatively compact in norm topology.

If in a Banach space X a partial order "$\leq$" is defined and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, then X is said to be a *Banach lattice.* If X is a Banach lattice and $\|x\| < \|y\|$ whenever $0 \leq x \leq y$ and $x \neq y$, then X is said to be *strictly monotone* (see [17]).

It is clear that a Banach space X is LUR if and only if it is CLUR and R (see [36]).

A Banach space X is said to have *property H* if on the unit sphere every weakly convergent sequence to a point on the sphere is convergent in norm.

A Banach space X has the *Schur property* if every weakly convergent sequence is norm convergent (strongly convergent) in X.

In this chapter we prove that for any reflexive Banach space X, both X and X^* are CLUR if and only if both X and X^* have property H. Criteria for rotundity, local uniform rotundity, compact local uniform rotundity and property H in Orlicz sequence spaces equipped with the Orlicz norm are given.

2. Main Results

We begin with a general result.

Theorem 2.1. If X is a reflexive Banach space, then both X and X^* are CLUR if and only if both X and X^* have property H.

Proof. It is known that if X is CLUR, then X has property H. We only need to prove that X is CLUR if both X and X^* have property H. For every $x_0 \in S(X)$ and every sequence (x_n) in $S(X)$ with $\lim_{n\to\infty} \| x_n + x_0 \| = 2$, take $(f_n) \subset S(X^*)$ such that $f_n(x_n + x_0) = \| x_n + x_0 \|$ for every $n \in \mathbb{N}$. Then

$$f_n(x_0) = \| x_n + x_0 \| - f_n(x_n)$$

for every $n \in \mathbb{N}$ and

$$\liminf_{n\to\infty} f_n(x_0) \geq \lim_{n\to\infty} \| x_n + x_0 \| - \limsup_{n\to\infty} f_n(x_n),$$

whence $\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} f_n(x_n) = 1$.

By the reflexivity of X, there is a subsequence (f_{n_i}) of (f_n) and $f_0 \in X^*$ such that f_{n_i} tends to f_0 weakly. It is obvious that in virtue of $\lim_{n\to\infty} f_n(x_0) = 1$ this yields $f_0(x_0) = 1$, whence $\|f_0\| = 1$. By property H for X^*, we get that $f_{n_i} \to f_0$ in norm. Hence

$$f_0(x_{n_i}) = (f_0 - f_{n_i})(x_{n_i}) + f_{n_i}(x_{n_i}) \to 1 \quad \text{as} \quad i \to \infty.$$

Using now the reflexivity of X, we can find a subsequence (z_i) of (x_{n_i}) and $x^0 \in X$ such that z_i tends to x^0 weakly. Obviously, $f_0(x^0) = 1$, whence $\| x^0 \| = 1$. By property H for X, z_i tends to x^0 strongly, i.e. the set $\{x_n \colon n \in \mathbb{N}\}$ is relatively compact in $S(X)$, which implies that X is CLUR.

Theorem 2.2. Let $x \in l_M^0 (x \neq 0)$. *If* $K(x) = \emptyset$, then

$$\| x \|_0 = \lim_{k \to \theta(x)^-} \frac{1}{k} (1 + \rho_M (kx)).$$

Proof. Since the function $f(k) = \frac{1}{k}(1 + \rho_M (kx))$ is continuous on the interval $(0, \theta(x))$ and $\lim_{k \to 0^+} f(k) = \infty$, the formula

$$\| x \|_0 = \lim_{k \to \theta(x)^-} \frac{1}{k} (1 + \rho_M (kx))$$

is true.

Corollary 2.3. If $x \in l_M^0$ and $\theta(x) < \infty$, then $K(x) \neq \emptyset$.

Proof. Assume for the contrary that $K(x) = \emptyset$. Then, by Theorem 2.2 and the Fatou Lemma, we have

$$\frac{1}{\theta(x)} (1 + \rho_M (\theta(x)x)) \leq \lim_{k \to \theta(x)^-} \frac{1}{k} (1 + \rho_M (kx)) = \| x \|_0 < \infty,$$

whence $K(x) \neq \emptyset$. A contradiction.

Corollary 2.4. If $x \in l_M^0$ *and* $K(x) = \emptyset$, then for any $n \in \mathbb{N}$ we have

$$\left\| \sum_{i=1}^{n} x(i) e_i \right\|_0 = \lim_{k \to \infty} \frac{1}{k} \rho_M \left(k \sum_{i=1}^{n} x(i) e_i \right)$$

and

$$\left\| \sum_{i=n+1}^{\infty} x(i) e_i \right\|_0 = \lim_{k \to \infty} \frac{1}{k} \rho_M \left(k \sum_{i=n+1}^{\infty} x(i) e_i \right).$$

Proof. We first claim that the limit $\lim_{u\to\infty}\dfrac{M(u)}{u}$ exists. This follows from the fact that for $0 < u_1 < u_2$, we have

$$\frac{M(u_1)}{u_1}=\frac{1}{u_1}M\left(\frac{u_1}{u_2}u_2\right)<\frac{u_1}{u_2}\cdot\frac{M(u_2)}{u_1}=\frac{M(u_2)}{u_2}.$$

Since $K(x)=\emptyset$, $\lim_{u\to\infty}\dfrac{M(u)}{u}$ is finite, and consequently

$$\begin{aligned}\|x\|_0 &= \lim_{k\to\infty}\frac{1}{k}\rho_M(kx)=\lim_{k\to\infty}\frac{1}{k}\left(\rho_M\left(k\sum_{i=1}^{n}x(i)e_i\right)+\rho_M\left(k\sum_{i=n+1}^{\infty}x(i)e_i\right)\right)\\ &= \lim_{k\to\infty}\frac{1}{k}\rho_M\left(k\sum_{i=1}^{n}x(i)e_i\right)+\lim_{k\to\infty}\frac{1}{k}\rho_M\left(k\sum_{i=n+1}^{\infty}x(i)e_i\right)\\ &\geq\left\|\sum_{i=1}^{n}x(i)e_i\right\|_0+\left\|\sum_{i=n+1}^{\infty}x(i)e_i\right\|_0\end{aligned}$$

for every $n\in\mathbb{N}$. On the other hand, by the triangle inequality, we have

$$\|x\|_0\leq\left\|\sum_{i=1}^{n}x(i)e_i\right\|_0+\left\|\sum_{i=n+1}^{\infty}x(i)e_i\right\|_0$$

for every $n\in\mathbb{N}$. So the corollary is proved.

Corollary 2.5. If $x\in l_M^0$ and $K(x)=\emptyset$, then

$$\|x\|_0=A\sum_{i=1}^{\infty}|x(i)|,$$

where $A = \lim_{u\to\infty} \dfrac{M(u)}{u}$.

Proof. Since we can assume without loss of generality that $x(i) \neq 0$ for any $i \in \mathbb{N}$, by Corollary 2.4, we have

$$\left\| \sum_{i=1}^{n} x(i)e_i \right\|_0 = \lim_{k\to\infty} \frac{1}{k} \rho_M\left(k\sum_{i=1}^{n} x(i)e_i \right)$$

$$= \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{n} M(kx(i)) = \lim_{k\to\infty} \sum_{i=1}^{n} |x(i)| \frac{M(kx(i))}{k|x(i)|} = A\sum_{i=1}^{n} |x(i)|$$

for every $n \in \mathbb{N}$. Since $\|x\|_0 = \lim_{n\to\infty} \left\| \sum_{i=1}^{n} x(i)e_i \right\|_0$, we obtain

$$\|x\|_0 = A\sum_{i=1}^{\infty} |x(i)|.$$

Corollary 2.6. Assume that M is an Orlicz function with

$$\lim_{u\to\infty} (M(u)/u) = A < \infty$$

and $a > 0$ is the number satisfying $M(a) = 1$. If there exists $b \in (0,1)$ such that $Au - b \leq M(u)$ for all $u \geq \frac{a}{2}$, then $K(x) = \emptyset$ for some $x \in S(l_M^0)$.

Proof. Define $y = (a,0,0\ldots)$. Then $\rho_M(y) = 1$, whence $\| y \| = 1$ and $\| y \|_0 \leq 2 \| y \| = 2$. So, for $x = \frac{1}{2}y$ there holds $\|x\|_0 \leq 1$. Assuming for the contrary that $K(x) \neq \emptyset$, we conclude that there is $k \geq 1$ such that

$$\| x \|_0 = \frac{1}{k}\left(1 + M\left(\frac{ak}{2}\right)\right)$$

$$\geq \frac{1}{k}\left(1 + \frac{Aak}{2} - b\right) = \frac{Aa}{2} + \frac{1-b}{k} > \frac{Aa}{2}.$$

On the other hand

$$\| x \|_0 \leq \lim_{k \to \infty} \frac{1}{k} M\left(\frac{ak}{2}\right)$$

$$= \frac{a}{2} \lim_{k \to \infty} \frac{2}{ak} M\left(\frac{ak}{2}\right) = \frac{Aa}{2},$$

a contradiction, which shows that $K(x) = \emptyset$.

Remark 2.7.

1. If M is an N'-function, then $K(x) \neq \emptyset$ for any $x \in l_M^0 \setminus \{0\}$.
2. Let M be an Orlicz function satisfying condition (1.5.1 of chapter-I). If $x \in l_M^0 \setminus \{0\}$ *and* $K(x) = \emptyset$, then supp x is a finite set.

Proof. The assertion (1) has been proved in [13]. To prove (2) assume for the contrary that supp x is infinite. Denote by μ the counting measure. Then

$$\mu\{i \in \mathbb{N} : |x(i)| > \frac{1}{n}\} \to \infty \quad \text{as} \quad n \to \infty.$$

By the assumption (1.5.1 of chapter-I), we conclude that both functions N and p vanish only at zero. Fix $a > 0$ and denote $b = p(a)$, where p is the right hand derivative of M on $\mathbb{R}_+$. Let $m\in\mathbb{N}$ satisfy $mN(b) \geq 1$. Next, let $n_0\in\mathbb{N}$ be such that

$$\mu\{i\in\mathbb{N}: |x(i)| > \frac{1}{n}\} \geq m$$

for $n \geq n_0$. Let $k = n_0 a$ and

$$A = \{i\in\mathbb{N}: |x(i)| > \frac{1}{n_0}\}.$$

Then $k\,|\,x(i)\,| > a$ and consequently $p(k\,|\,x(i)\,|) \geq b$ for any $i\in A$. Therefore,

$$\rho_N(pok\,|\,x\,|) \geq \mu(A)N(b) \geq mN(b) \geq 1.$$

This yields

$$k_x^* = \inf\{k > 0 : \rho_N(pok\,|\,x\,|) \geq 1\} < \infty,$$

whence $K(x) \neq \emptyset$ (see[1]).

Lemma 2.8. Let $x_0 \in S(l_M^0)$ be such that $K(x_0)$ is nonempty and bounded. If $(x_n) \subset S(l_M^0)$ is coordinatewise convergent to x_0 then there exists $n_0\in\mathbb{N}$ such that $K(x_n) \neq \emptyset$ for all $n \geq n_0$ and $\sup_{n\geq n_0}\{k_n\} < \infty$ for any sequence (k_n) with $k_n \in K(x_n)$ $(n\in \mathbb{N})$. If additionally $M\in\delta_2$, then there exists a subsequence $(x_{n_k})$of (x_n) such that all elements of (x_{n_k}) have equi-absolutely continuous norms, i.e. for any $\varepsilon > 0$ there is $i_\varepsilon\in\mathbb{N}$ such that

$$\left\| \sum_{i=i_\varepsilon}^{n} x_{n_k}(i) e_i \right\|_0 < \varepsilon$$

for all $k \in \mathbb{N}$.

Proof. Suppose that $K(x_0)$ is nonempty and bounded. First, we will prove that there is $n_1 \in \mathbb{N}$ such that $K(x_n) \neq \emptyset$ for $n \geq n_1$. Otherwise, we may assume without loss of generality that $K(x_n) = \emptyset$ $(n = 1,2,\ldots)$. This implies that $\lim_{u\to\infty}(M(u)/u) = A < \infty$ because otherwise $K(x) \neq \emptyset$ for any $x \in l_M^0$ (see[13]). Since $K(x_0) \neq \emptyset$ and it is bounded, there is $\varepsilon_0 > 0$ such that

$$\| x_0 \|_0 + 2\varepsilon_0 < \lim_{k\to\infty} \frac{1}{k} \rho_M(kx_0) = A \sum_{i=1}^{\infty} |x_0(i)|.$$

Next, there exists $i_1 \in \mathbb{N}$ such that

$$\| x_0 \|_0 + \varepsilon_0 < A \sum_{i=1}^{i_1} |x_0(i)|.$$

Since $x_n \to x_0$ coordinatewise, there is $n_2 \in \mathbb{N}$ such that

$$\sum_{i=1}^{i_1} |x_n(i)| > \sum_{i=1}^{i_1} |x_n(i)| - \frac{\varepsilon_0}{2A}$$

for $n \geq n_2$. Hence

$$1 = \| x_n \|_0 = \lim_{k\to\infty} \frac{1}{k} \rho_M(kx_n)$$

$$= A\sum_{i=1}^{\infty} |x_n(i)| = A\sum_{i=1}^{i_1} |x_n(i)| + A\sum_{i=i_1+1}^{\infty} |x_n(i)|$$

$$> A\left(\sum_{i=1}^{i_1} |x_0(i)| - \frac{\varepsilon_0}{2A}\right) + A\sum_{i=i_1+1}^{\infty} |x_n(i)| = A\sum_{i=1}^{i_1} |x_0(i)| - \frac{\varepsilon_0}{2} + A\sum_{i=i_1+1}^{\infty} |x_n(i)|$$

$$> \| x_0 \|_0 + \varepsilon_0 - \frac{\varepsilon_0}{2} + A\sum_{i=i_1+1}^{\infty} |x_n(i)| \geq 1 + \frac{\varepsilon_0}{2}$$

for $n \geq n_2$. This contradiction shows that there is $n_1 \in \mathbb{N}$ such that $K(x_n) \neq \emptyset$ for $n \geq n_1$. Without loss of generality, we may assume that $K(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, there are $k_n \geq 1$ such that

$$\| x_n \|_0 = \frac{1}{k_n}(1 + \rho_M(k_n x_n)), \qquad n = 0,1,2,...$$

We will prove that $\sup\{k_n : n = 0,1,2,....\} < \infty$. If not, we can assume that $k_n \uparrow \infty$. Since

$$\| x_0 \|_0 + 2\varepsilon_0 < \lim_{k\to\infty} \frac{1}{k}\rho_M(kx_0) = A\sum_{i=1}^{\infty} |x_0(i)|,$$

there exists $i_2 \in \mathbb{N}$ such that

$$\| x_0 \|_0 + \varepsilon_0 < A\sum_{i=1}^{i_2} |x_0(i)|.$$

Hence

$$1 = \| x_n \|_0 = \frac{1}{k_n}(1 + \rho_M(k_n x_n))$$

$$= \frac{1}{k_n}\left(1 + \sum_{i=1}^{\infty} M(k_n x_n(i))\right) = \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_2} M(k_n x_n(i)) + \sum_{i=i_2+1}^{\infty} M(k_n x_n(i))\right)$$

$$\geq \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_2} M(k_n x_n(i))\right) \to A \sum_{i=1}^{i_2} |x_0(i)| > 1 + \varepsilon_0,$$

a contradiction which shows that $\sup\{k_n : n = 0,1,2,\ldots\} < \infty$ whenever $K(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Taking n_0 large enough and using the fact that $K(x_n) \neq \emptyset$ for all $n \geq n_1$, we get the first part of the lemma.

Now, suppose that $M \in \delta_2$. Passing to a subsequence if necessary, we may assume without loss of generality that there is $k \in [1,\infty)$ such that

$$\lim_{n \to \infty} k_n = k.$$

Now, we will prove that the elements of (x_n) have equi-absolutely continuous norms. Since $M \in \delta_2$, we only need to prove that for every $\varepsilon > 0$ there is $i_\varepsilon \in \mathbb{N}$ such that

$$\rho_M\left(\sum_{i=i_\varepsilon}^{\infty} x_n(i) e_i\right) < \varepsilon$$

for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there is $i_3 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{i_3} x_0(i) e_i \right\|_0 > 1 - \frac{\varepsilon}{2}.$$

Since $x_n \to x_0$ coordinatewise, there is $n_3 \in \mathbb{N}$ such that

$$\frac{1}{k_n}\left(1+\rho_M\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right)\right) \geq \frac{1}{k}\left(1+\rho_M\left(\sum_{i=1}^{i_3} kx_0(i)e_i\right)\right)-\frac{\varepsilon}{2}$$

for $n \geq n_3$. Hence

$$1=\frac{1}{k_n}(1+\rho_M(k_n x_n))=\frac{1}{k_n}\left(1+\rho_M\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right)+\rho_M\left(\sum_{i=i_3+1}^{\infty} k_n x_n(i)e_i\right)\right)$$

$$\geq \frac{1}{k_n}\left(1+\rho_M\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right)\right)+\rho_M\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)$$

$$\geq \frac{1}{k}\left(1+\rho_M\left(\sum_{i=1}^{i_3} kx_0(i)e_i\right)\right)-\frac{\varepsilon}{2}+\rho_M\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)$$

$$\geq \left\|\sum_{i=1}^{i_3} x_0(i)e_i\right\|_0-\frac{\varepsilon}{2}+\rho_M\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)>1-\varepsilon+\rho_M\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)$$

for $n \geq n_3$. So we have

$$\rho_M\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)<\varepsilon$$

for $n \geq n_3$. This finishes the proof of the lemma.

Theorem 2.9. If $M \in \delta_2$, then for each sequence (x_n) in $S(l_M^0)$ *and* $x_0 \in S(l_M^0)$ such that $x_n(i) \to x_0(i)$ *as* $n \to \infty$ for $i = 1,2,3\ldots,$ we have that $x_n \to x_0$ in norm.

Proof. Note that $M \in \delta_2$ yields that M vanishes only at zero. In order to prove the theorem, we will consider two cases.

Case I. $K(x_n) = \emptyset$ or $K(x_0)$ is nonempty and unbounded. Assume first that there is a subsequence (z_n) of (x_n) such that $K(z_n) = \emptyset$ for all $n \in \mathbb{N}$. Then, by Corollary 2.5, $\| z_n \|_0 = A\sum_{i=1}^{\infty} |z_n(i)|$ for each $n \in \mathbb{N}$ and $\| x_0 \|_0 = A\sum_{i=1}^{\infty} |x_0(i)|$. Since l_1 has the Schur property, we get

$$A\sum_{i=1}^{\infty} |z_n(i) - x_0(i)| \to 0 \quad \text{as} \quad n \to \infty.$$

Since

$$\| z_n - x_0 \|_0 \leq A\sum_{i=1}^{\infty} |z_n(i) - x_0(i)|$$

for all $n \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \| z_n - x_0 \|_0 = 0.$$

Assume now that there is a subsequence (y_n) of (x_n) such that there are $k_n \geq 1$ satisfying

$$\| y_n \|_0 = \frac{1}{k_n}(1 + \rho_M(k_n y_n))$$

for $n = 1,2,\ldots$. In virtue of $M \in \delta_2$, for any $\varepsilon > 0$ there is $\delta > 0$ such that $\rho_M(x) < \delta$ implies $\| x \|_0 < \varepsilon$. Moreover, there is $i_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{i_0} x_0(i) e_i \right\|_0 > 1 - \delta$$

and

$$\left\| \sum_{i=i_0+1}^{\infty} x_0(i)e_i \right\|_0 < \varepsilon.$$

Since $y_n(i) \to x_0(i)$ as $n\to\infty$ for $i=1,2,\ldots$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{i_0} y_n(i)e_i \right\|_0 > 1-\delta$$

for $n \geq n_0$. So,

$$1 = \frac{1}{k_n}(1+\rho_M(k_n y_n)) = \frac{1}{k_n}\left(1+\sum_{i=1}^{i_0} M(k_n y_n(i)) + \sum_{i=i_0+1}^{\infty} M(k_n y_n(i))\right)$$

$$\geq \frac{1}{k_n}\left(1+\sum_{i=1}^{i_0} M(k_n y_n(i))\right) + \sum_{i=i_0+1}^{\infty} M(y_n(i))$$

$$\geq \left\|\sum_{i=1}^{i_0} y_n(i)e_i\right\|_0 + \sum_{i=i_0+1}^{\infty} M(y_n(i)) > 1-\delta + \sum_{i=i_0+1}^{\infty} M(y_n(i))$$

for $n \geq n_0$. This means that

$$\sum_{i=i_0+1}^{\infty} M(y_n(i)) < \delta$$

for $n \geq n_0$. Hence

$$\left\| \sum_{i=i_0+1}^{\infty} y_n(i)e_i \right\|_0 < \varepsilon$$

for $n \geq n_0$. Since $y_n \to x_0$ coordinatewise, there is $n_1 \geq n_0$ such that

$$\left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i))e_i \right\|_0 < \varepsilon$$

for $n \geq n_1$. Thus

$$\| y_n - x_0 \|_0 = \left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i))e_i + \sum_{i=i_0+1}^{\infty} y_n(i)e_i - \sum_{i=i_0+1}^{\infty} x_0(i)e_i \right\|_0$$

$$\leq \left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i)e_i \right\|_0 + \left\| \sum_{i=i_0+1}^{\infty} y_n(i)e_i \right\|_0 + \left\| \sum_{i=i_0+1}^{\infty} x_0(i)e_i \right\|_0 < 3\varepsilon$$

for $n \geq n_1$. Note that we have proved that we always can find a subsequence (y_n) of (x_n) such that $\| y_n - x_0 \| \to 0$. So, by the double extract subsequence theorem, there holds $\| x_n - x_0 \| \to 0$.

Case II. $K(x_0)$ is nonempty and bounded. By Lemma 2.8, we can assume without loss of generality that $K(x_n) \neq \emptyset$ for each $n \in \mathbb{N}$. Now, repeating the procedure from case I, we get

$$\lim_{n \to \infty} \| x_n - x_0 \|_0 = 0.$$

We can also get the same conclusion by applying equi-absolute continuity of the norm of (x_n) and the fact that $x_n \to x_0$ coordinatewise. So the proof of the theorem is finished.

Remark 2.10. An analogue of Theorem 2.9 for the Luxemburg norm has been proved in [18].

Remark 2.11. Criteria for H-property of Orlicz sequence spaces equipped with the Luxemburg norm and the Orlicz norm, but generated by N-functions, were first given in [45] and [3]. Next the problem of a characterization of H-points in Orlicz sequence spaces was considered in [15], [38] and [18]. Criteria for H-property of Orlicz sequence spaces l_M equipped with the Luxemburg norm in the case of convex Orlicz functions (without the assumption that it is an N-function) were given in [18]. The next theorem solves an analogous problem for the Orlicz norm for arbitrary Orlicz function. Although the criterion is the same as for N-functions, the proof is much more complicated and it is based on Theorem 2.9, the proof of which use some new techniques in comparison with the ones used in [45] and [3].

3. Some Geometric Properties

In this section we will consider some geometric properties of the space l_M^0.

Theorem 3.1. The space l_M^0 has property H if and only if $M \in \delta_2$.

Proof. *Sufficiency.* Assume for a sequence (x_n) in $S(l_M^0)$ that $x_n \to x_0$ weakly. This implies that $x_n \to x_0$ coordinatewise. By Theorem 2.9, in view of $M \in \delta_2$, $x_n \to x_0$ in norm.

Necessity. Assume first that M vanishes only at zero. If $M \notin \delta_2$, there is $x_0 \in S(l_M)$ such that

$$\rho_M(x_0) \le 1 \text{ and } \rho_M(\lambda x_0) = \infty$$

for any $\lambda > 1$ (see [1]). Take an increasing sequence (i_n) of natural numbers such that $i_n \to \infty$ as $n \to \infty$ and

$$\left\| \sum_{i=i_n+1}^{i_{n+1}} x_0(i) e_i \right\|_0 \ge \frac{1}{2}$$

for every $n \in \mathbb{N}$. Put

$$x_n = (x_0(1), \ldots, x_0(i_n), 0, \ldots 0, x_0(i_{n+1}+1), x_0(i_{n+1}+2), \ldots)$$

and $z_n = x_n / \| x_0 \|_0$ for $n = 0, 1, 2, \ldots$.

We will show that $z_n \to z_0$ weakly. Any $f \in \left(l_M^0\right)^*$ is uniquely represented in the form $f = y + s$, where $y \in l_N$ and $s \in \left(h_M^0\right)^\perp$. Since $y \in l_N$, there is $\lambda > 0$ such that

$$\sum_{i=1}^{\infty} N(\lambda y(i)) < \infty .$$

Moreover, since $x_n - x_0 \in h_M^0$, we have $\langle x_n - x_0, s \rangle = 0$ for each $n \in \mathbb{N}$. Hence

$$|\langle x_n - x_0, f \rangle| = |\langle x_n - x_0, y \rangle| = \left| \sum_{i=i_n+1}^{i_{n+1}} x_0(i) y(i) \right|$$

$$\le \frac{1}{\lambda} \left(\sum_{i=i_n+1}^{i_{n+1}} M(x_0(i)) + N(\lambda y(i)) \right) \to 0$$

as $n\to\infty$. So, $z_n \to z_0$ weakly. Hence, by property H of l_M^0,

$$\| z_n - z_0 \|_0 \to 0.$$

On the other hand,

$$\| z_n - z_0 \|_0 = \frac{1}{\| x_0 \|_0} \left\| \sum_{i=i_n+1}^{i_{n+1}} x_0(i) e_i \right\|_0 \geq \frac{1}{2 \| x_0 \|_0} > 0$$

for every $n \in \mathbf{N}$. This contradiction shows that $M \in \delta_2$ is necessary for property H of l_M^0 if M vanishes only at zero.

Assume now that M vanishes outside zero and define

$$a = a(M) = \sup\{u \geq 0 : M(u) = 0\}.$$

Then $a > 0$. Let $x = (a, a, \ldots)$. We have $1 + \rho_M(x) = 1$ and

$$\frac{1}{k}\left(1 + \rho_M(kx)\right) > 1$$

for each $k > 0$, $k \neq 1$. Therefore $\| x \|_0 = 1$. Define

$$x_n = (\underbrace{a, \ldots, a}_{n}, 0, a, a, \ldots)$$

for every $n \in \mathbf{N}$. Then we can prove in the same way as for x that $\| x_n \|_0 = 1$ for each $n \in \mathbf{N}$. Since $x - x_n = a e_{n+1} \in h_M$, $x^*(x - x_n) = 0$ for each singular functional x^* over l_M. Take now any $y = (y_1, y_2, \ldots) \in l_N^0$. There is $\lambda > 0$ such that $\rho_N(\lambda y) < \infty$, whence it follows that $N(\lambda y_n) \to 0$

as $n \to \infty$. Therefore, since N vanishes only at zero, we get $y_n \to 0$ as $n \to \infty$ and consequently

$$\langle x - x_n, y \rangle = a y_{n+1} \to 0 \text{ as } n \to \infty .$$

This shows that $x - x_n \to 0$ weakly. However

$$\| x - x_n \| = a \| e_{n+1} \| = a \| e_1 \| > 0$$

for each $n \in \mathbb{N}$. So, l_M^0 fails to have property H if M vanishes outside zero, which finishes the proof of the theorem.

Theorem 3.2. The space l_M^0 is CLUR if and only if $M \in \delta_2$ and $N \in \delta_2$.

Proof. *Sufficiency.* If $M \in \delta_2$ and $N \in \delta_2$, then l_M^0 is reflexive. Consequently, by the previous theorem and Theorem 6 in [18], both l_M^0 and l_N have property H. In virtue of Theorem 2.1, we obtain that l_M^0 is CLUR.

Necessity. Since property CLUR implies property H, in view of Theorem 3.1, we get $M \in \delta_2$. To prove the necessity of $N \in \delta_2$ assume first that N vanishes only at zero. If $N \notin \delta_2$, there is a sequence (u_n) of positive numbers such that $u_n \downarrow 0$ and

$$N\left(\left(1 + \frac{1}{n}\right) u_n\right) \geq 2^n N(u_n) \text{ and } N(u_n) \leq \frac{1}{2^{n-1}} \tag{3.2.1}$$

for each $n \in \mathbb{N}$. Take positive integers B_m such that

$$\frac{1}{2^m} < B_m \, N(u_m) \leq \frac{1}{2^{m-1}} \tag{3.2.2}$$

for $m = 1,2,\ldots$ (we can pass to a subsequence of (u_n) if necessary).

Define $k_i = \sum_{m=1}^{i} B_m \quad (i = 1,2,\ldots)$,

$b(N) = \sup\{u \geq 0 \colon N(\mathrm{u}) \leq 1\}$ and

$z_0 = (b(N),0,0,\ldots)$,

$$z_1 = (\overbrace{u_1, u_1, \ldots, u_1}^{B_1}, 0,0,\ldots),$$

$$z_2 = (\overbrace{0,0,\ldots,0,}^{B_1}\ \overbrace{u_2, u_2, \ldots, u_2}^{B_2}, 0,0,\ldots),$$

....

$$z_m = (\overbrace{0,0,\ldots,0,}^{k_{m-1}}\ \overbrace{u_m, u_m, \ldots, u_m}^{B_m}, 0,0,\ldots),$$

....

Then, denoting by $\|\cdot\|$ the Luxemburg norm in l_N, we easily get from inequalities (3.2.1) and (3.2.2) that

(i) $\dfrac{m}{m+1} \leq \| z_m \| \leq 1, m = 1,2,\ldots$

Moreover,

(ii) there is a sequence (x_m) in $S\left(l_M^0\right)$ such that $x_m(i) = 0$ for $1 \leq i \leq k_{m-1}$ and $i \geq k_m+1$, and x_m generate support functionals at z_m, i.e.

$$\| z_m \| = \langle z_m, x_m \rangle = u_m \sum_{i=k_{m-1}+1}^{k_m} x_m(i).$$

This follows by the Hahn-Banach theorem and the fact that $(l_N)^* = l_M^0$.

Moreover for

$$x_0 = \left(\frac{1}{b(N)}, 0, 0 \ldots \right)$$

there holds $\|x_0\|_0 = 1$. Put

$$g_m = \left(1 - \frac{1}{2^{m-1}} \right) (\overbrace{b(N), 0, 0, \ldots 0}^{k_m}, z_m(k_{m-1}+1), z_m(k_{m-1}+2), \ldots, z_m(k_m), 0, 0, \ldots)$$

for $m = 1, 2, \ldots$. Then

$$\rho_N(g_m) \leq \left(1 - \frac{1}{2^{m-1}} \right) \left(1 + \frac{1}{2^{m-1}} \right) = 1 - \frac{1}{4^{m-1}} \leq 1,$$

whence $\|g_m\| \leq 1$. Moreover, in virtue of (ii), we have

$$\| x_m + x_0 \|_0 \geq \langle x_m + x_0, g_m \rangle = \sum_{i=1}^{\infty} (x_m(i) + x_0(i)) g_m(i)$$

$$= \left(1 - \frac{1}{2^{m-1}} \right) \left(1 + u_m \sum_{i=k_{m-1}+1}^{k_m} x_m(i) \right) \to 2$$

as $m \to \infty$. But, by the orthogonality of x_m and x_n for $n \neq m$, there holds

$$\| x_m - x_n \|_0 \geq \| x_n \|_0 = 1 \text{ for } n \neq m,$$

which means that l_M^0 is not CLUR. So, in the case when N vanishes only at zero, $N \in \delta_2$ is necessary for property CLUR of l_M^0.

Assume now that

$$c(N) = \sup\{u \geq 0: N(u) = 0\} > 0$$

and define

$$u_n = \left(1 - \frac{1}{n^2}\right) c(N) \text{ and } \lambda_n = \frac{n}{n-1}$$

for $n = 2,3,\ldots$. Then

$$\lambda_n u_n = \frac{n^2 - 1}{n^2 - n} c(N) > c(N)$$

for each $n \in \mathbb{N}\setminus\{1\}$, but $\lambda_n u_n \to c(N)$ as $n \to \infty$. Let $(B_n)_{n=1}^{\infty}$ be a sequence of natural numbers such that

$$B_{n-1} N(\lambda_n u_n) \geq 1$$

for $n = 2,3,\ldots$. Define

$$z_1 = \sum_{i=2}^{B_1+1} u_2 e_i$$

and

$$z_n = \sum_{i=k_{n-1}+1}^{k_n} u_{n+1} e_i,$$

where $k_n = 1 + \sum_{i=1}^{n-1} B_i$ for $n = 2,3,\ldots$. Define also $z_0 = (b(N),0,0,\ldots)$, where $b(N)$ is the number defined above. Then $\| z_0 \| = 1$ and

$$\rho_N(z_n) = 0, \quad \rho_N(\lambda_n z_n) \geq 1$$

for each $n \in \mathbb{N}$. Hence

$$\lambda_n^{-1} \leq \| z_n \| \leq 1$$

for each $n \in \mathbb{N}$. Since $z_n \in h_N$ for every $n \in \mathbb{N} \cup \{0\}$, there is a sequence $(x_n)_{n=0}^{\infty}$ with supp x_n = supp z_n such that

$$\| x_n \|_0 = 1 \text{ and } \langle z_n, x_n \rangle = \| z_n \|$$

for each $n \in \mathbb{N} \cup \{0\}$. Define $g_n = z_n + z_0$ for each $n \in \mathbb{N}$. Then

$$\rho_N (g_n) = \rho_N (z_n) + \rho_N (z_0) = \rho_N (z_0) \leq 1,$$

whence $\| g_n \| \leq 1$. Moreover, $\| g_n \| \geq \| z_0 \| = 1$ and consequently $\| g_n \| = 1$ for each $n \in \mathbb{N}$. Hence

$$2 \geq \| x_n + x_0 \|_0 \geq \langle x_n + x_0, g_n \rangle = \| x_n \|_0 + \| x_0 \|_0 \rightarrow 2,$$

whence

$$\lim_{n \to \infty} \| x_n + x_0 \|_0 = 2.$$

Since

$$\| x_n - x_0 \|_0 \geq \| x_n \|_0 = 1$$

for every $n \neq m$, the sequence (x_n) contains no convergent sequence, i.e. l_M^0 is not CLUR.

This finishes the proof of the theorem.

CHAPTER-III

ORLICZ VECTOR VALUED SEQUENCE SPACES

1. Preliminaries and Introduction

A point $x \in S(X)$ is called an *extreme point* of $B(X)$ if for every $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$. We denote Ext $B(X)$ the set of all extreme points of $B(X)$. A Banach space X is said to be *rotund* (write (R) for short), if Ext $B(X) = S(X)$ is an extreme point.

Let X be a Banach space. Denote X^0 the space of all sequences in X. For $x \in X^0$, we denote $x(i)$ the i^{th} term of x.

For a given Orlicz function M, we define $\rho_M : X^0 \to [0,\infty]$ by the formular

$$\rho_M(x) = \sum_{i=1}^{\infty} M(\| x(i) \|).$$

By convexity of M it is easy to see that ρ_M is a convex modular.

The Orlicz vector-valued sequence space $l_M(X)$ and its subspace $h_M(X)$ are defined as follows:

$$l_M(X) := \{x \in X^0 : \rho_M(cx) < \infty \quad \text{for some } c > 0\},$$

$$h_M(X) := \{x \in l_M(X) : \rho_M(cx) < \infty \quad \text{for all } c > 0\}.$$

We consider $l_M(X)$ equipped with the Luxemburg norm

$$\| x \|_M = \inf\{\lambda > 0 : \rho_M (x/\lambda) \leq 1\}.$$

Since ρ_M is a convex modular, it implies that $\left(l_M, \|\cdot\|_M\right)$ is a Banach space (see [35]).

Convexity properties in Banach space is an important topic in functional analysis and play an important role in infinite dimensional holomorphy. In order to study the geometric properties of Banach space, Clarkson [4] introduced the very important class of rotundity (strict convexity). Since Clarkson's paper many authors have defined and studied the classes of Banach space lying between the uniform convexity and rotundity see [30] and [43]. The criteria for rotundity of Orlicz spaces was given by [2].

2. Main Results

In order to establish our main results, we start with giving some auxiliary lemmas.

Lemma 2.1. Suppose that ρ is a convex modular on a real vector space X such that $\rho \in \delta_2^s$. Then for $x \in X_\rho$, $\|x\| = 1$ if and only if $\rho(x) = 1$

Proof. See [23].

Lemma 2.2. If $M \notin \delta_2$, then there exists $x \in l_M(X)$ such that $\|x\|_M=1$ and $\rho_M(x) < 1$.

Proof. Let $t > 1$. If $M \notin \delta_2$, by Theorem 1.7.2(2) of chapter I, there exists $\alpha_k \downarrow 0$ such that $M(\alpha_k) < \frac{1}{2^{k+1}}$ and $M(t\alpha_k) > 2^{k+1} M(\alpha_k)$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose an integer m_k such that

$$\frac{1}{2^{k+1}} \leq m_k M(\alpha_k) < \frac{1}{2^k}.$$

Put $x' \in S(X)$ and define

$$x = (\overbrace{\alpha_1 x', \alpha_1 x', \ldots, \alpha_1 x'}^{m_1 times}, \overbrace{\alpha_2 x', \alpha_2 x', \ldots, \alpha_2 x'}^{m_2 times}, \ldots, \overbrace{\alpha_k x', \alpha_k x', \ldots, \alpha_k x'}^{m_k times}, \ldots).$$

Observe that

$$\rho_M(x) = \sum_{i=1}^{m_1} M(\| \alpha_1 x(i) \|) + \sum_{i=m_1+1}^{m_1+m_2} M(\| \alpha_2 x(i) \|) + \sum_{i=m_1+m_2+1}^{m_1+m_2+m_3} M(\| \alpha_3 x(i) \|) + \ldots$$

$$+ \sum_{i=m_1+m_2+\ldots+m_{k-1}+1}^{m_1+m_2+\ldots+m_k} M(\| \alpha_k x(i) \|) + \ldots$$

$$= m_1 M(\alpha_1) + m_2 M(\alpha_2) + \ldots + m_k M(\alpha_k) + \ldots$$

$$= \sum_{k=1}^{\infty} m_k M(\alpha_k) < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

This show that $x \in l_M(X)$ and $\rho_M(x) < 1$.

But we see that

$$\rho_M(tx) = \sum_{k=1}^{\infty} m_k M(t\alpha_k) > \sum_{k=1}^{\infty} m_k 2^{k+1} M(\alpha_k) \geq \sum_{k=1}^{\infty} 1 = \infty,$$

which implies that $\|x\|_M = 1$.

Theorem 2.3. $x \in S(l_M(X))$ is an extreme point of $B(l_M(X))$ if and only if

(1) $\rho_M(x) = 1$,

(2) $\dfrac{x(i)}{\| x(i) \|} \in$ Ext $B(X)$ for all $i \in \mathbb{N}$ such that $x(i) \neq 0$,

(3) $\#\{i : \| x(i) \| \in \mathbb{R} \backslash S_M\} \leq 1$.

Where, for a finite subset $A \subseteq \mathbb{N}$ we denoted $\#A$ the number of member of A.

Proof. *Necessity.* Suppose (1) does not hold, i.e., $\rho_M(x) = c < 1$. Since M is continuous, we can choose $\varepsilon > 0$ so small such that

$$M(\| x(1) \| + \varepsilon) \leq M(\| x(1) \|) + \frac{1-c}{2}.$$

By $\rho_M(x) = \sum_{i=1}^{\infty} M(\| x(i) \|) < 1$, we have $\lim_{i \to \infty} M(\| x(i) \|) = 0$. This implies $\| x(i) \| \to 0$ as $i \to \infty$. So there exists $K \in \mathbb{N}$ such that $\| x(n) \| < \varepsilon$ for all $n \geq K$.

Next, we select $x(n_0)$ for some $n_0 \geq K$ and defined $y = (y(i))$, $z = (z(i))$ by $y(1) = x(1) - x(n_0)$, $z(1) = x(1) + x(n_0)$ and $y(i) = x(i) = z(i)$ for all $i \geq 2$. Then $2x = y + z$ and $y \neq z$. Then

$$\rho_M(y) = M(\| x(1) - x(n_0) \|) + \sum_{i=2}^{\infty} M(\| x(i) \|)$$

$$\leq M(\| x(1) \| + \| x(n_0) \|) + \sum_{i=2}^{\infty} M(\| x(i) \|)$$

$$\leq M(\|x(1)\|+\varepsilon)+\sum_{i=2}^{\infty} M(\|x(i)\|)$$

$$\leq M(\|x(1)\|)+\frac{1-c}{2}+\sum_{i=2}^{\infty} M(\|x(i)\|)$$

$$=\rho_M(x)+\frac{1-c}{2}$$

$$=c+\frac{1-c}{2}=\frac{c+1}{2}<1,$$

and similarly we can show that $\rho_M(z)\leq 1$. Thus $y,z\in B(l_M(X))$ which contradicts with the fact that $x\in\text{Ext } B(l_M(X))$.

If (2) is not true, then there exist $i_0\in\mathbf{N}$ such that $\frac{x(i_0)}{\|x(i_0)\|}\notin\text{Ext } B(X)$. Then there exist $u(i_0),v(i_0)\in B(X)$ with $u(i_0)\neq v(i_0)$ and $\frac{x(i_0)}{\|x(i_0)\|}=\frac{u(i_0)+v(i_0)}{2}$. Define the sequence u' and v' by,

$$u'(i)=\begin{cases}\|x(i_0)\|\,u(i_0) & ;i=i_0\\ x(i) & ;i\neq i_0,\end{cases}$$

and

$$v'(i)=\begin{cases}\|x(i_0)\|\,v(i_0) & ;i=i_0\\ x(i) & ;i\neq i_0.\end{cases}$$

It is easy to see that $\rho_M(u')\leq 1$ and $\rho_M(v')\leq 1$, $u'\neq v'$ and $2x=u'+v'$ which contradicts our hypothesis that x is an extreme point of $B(X)$.

Next, assume that (3) does not holds, without loss of generality we may assume that $\| x(1) \|, \| x(2) \| \in \mathbb{R} \backslash S_M$ i.e., $\| x(1) \|$ and $\| x(2) \|$ belong to some affine intervals (a_1,b_1) and (a_2,b_2) of M, respectively.

Let $M(u) = k_i u + \beta_i$, $u \in (a_i, b_i)$ $(i = 1,2)$. Select $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $k_1 \varepsilon_1 \| x(1) \| = k_2 \varepsilon_2 \| x(2) \|$ and $(1 \pm \varepsilon_i) \| x(i) \| \in (a_i, b_i)$ $(i = 1,2)$.

Define $y = (y\,(i))$, $z = (z\,(i))$ by

$$y(1) = (1+\varepsilon_1)x(1) \qquad y(2) = (1-\varepsilon_2)x(2)$$

$$z(1) = (1-\varepsilon_1)x(1) \qquad z(2) = (1+\varepsilon_2)x(2)$$

and $y(i) = z(i) = x(i)$ for all $i > 2$. So

$$\begin{aligned} y(1) + z(1) &= (1+\varepsilon_1)x(1) + (1-\varepsilon_1)x(1) \\ &= x(1) + \varepsilon_1 x(1) + x(1) - \varepsilon_1 x(1) \\ &= 2x(1) \end{aligned}$$

$$\begin{aligned} y(2) + z(2) &= (1-\varepsilon_2)x(2) + (1+\varepsilon_2)x(2) \\ &= x(2) - \varepsilon_2 x(2) + x(2) + \varepsilon_2 x(2) \\ &= 2x(2) \end{aligned}$$

and $y(i) + z(i) = 2x(i)$ for all $i > 2$.

Hence $2x = y + z$ and $y \neq z$. Now, we show that y and $z \in B(l_M(X))$. By computation $\rho_M(y)$, we have

$$\rho_M(y) = M(\| y(1) \|) + M(\| y(2) \|) + \sum_{i=3}^{\infty} M(\| y(i) \|)$$

$$= M(\| (1+\varepsilon_1)x(1) \|) + M(\| (1-\varepsilon_2)x(2) \|) + \sum_{i=3}^{\infty} M(\| x(i) \|)$$

$$= k_1((1+\varepsilon_1)\| x(1) \|) + \beta_1 + k_2((1-\varepsilon_2)\| x(2) \|) + \beta_2 + \sum_{i=3}^{\infty} M(\| x(i) \|)$$

$$= k_1 \| x(1) \| + k_1\varepsilon_1 \| x(1) \| + \beta_1 + k_2 \| x(2) \| - k_2\varepsilon_2 \| x(2) \| + \beta_2 + \sum_{i=3}^{\infty} M(\| x(i) \|)$$

$$= k_1 \| x(1) \| + \beta_1 + k_2 \| x(2) \| + \beta_2 + \sum_{i=3}^{\infty} M(\| x(i) \|)$$

$$= M(\| x(1) \|) + M(\| x(2) \|) + \sum_{i=3}^{\infty} M(\| x(i) \|)$$

$$= \rho_M(x) \le 1.$$

Similarly, we get that $\rho_M(z) \le 1$, hence $y, z \in B(l_M(X))$ which contradicts our hypothesis that $x \in \operatorname{Ext} B(l_M(X))$.

Sufficiency. Let $2x = y + z$, $y, z \in B(l_M(X))$. Since

$$1 = \rho_M(x) = \rho_M\left(\frac{y+z}{2}\right) \le \frac{1}{2}[\, \rho_M(y) + \rho_M(z)\,] \le 1,$$

we have

$$M\left(\frac{\| y(i) \| + \| z(i) \|}{2}\right) = \frac{1}{2}\left[M(\| y(i) \|) + M(\| z(i) \|)\right],$$

for all $i \in \mathbb{N}$. By (3) there exist atmost one $j \in \mathbb{N}$ such that $\|x(j)\| \in \mathbb{R} \setminus S_M$. This give $\|x(i)\| = \|y(i)\| = \|z(i)\|$ for all $i \ne j$. And since

$$1 = \sum_{i=1}^{\infty} M(\| x(i) \|) = \sum_{i=1}^{\infty} M(\| y(i) \|) = \sum_{i=1}^{\infty} M(\| z(i) \|),$$

we deduce $\|x(j)\| = \|y(j)\| = \|z(j)\|$. Since $2(x(i)) = y(i) + z(i)$ we have $\frac{2x(i)}{\|x(i)\|} = \frac{y(i)}{\|y(i)\|} + \frac{z(i)}{\|z(i)\|}$. It implies by (2) that $y = z$. Hence, $x \in \text{Ext } B(l_M(X))$.

Theorem 2.4. $l_M(X)$ is rotund if and only if

1. $M \in \delta_2$,
2. X is rotund and
3. M is strictly convex on $[0, M^{-1}(1/2)]$.

Proof. *Necessity.* If $M \notin \delta_2$, then by Lemma 2.2 we can find $x \in l_M(X)$ such that $\|x\|_M = 1$ and $\rho_M(x) < 1$, thus by (1) of Theorem 2.3 we have that $x \notin \text{Ext } B(l_M(X))$.

If (2) is not true, then there exist $x, y, z \in S(X)$ with $2x = y + z$ and $y \neq z$. Pick $u \in S(l_M(X))$, by (1) and Lemma 2.1 we have $1 = \rho_M(u) = \sum_{i=1}^{\infty} M(\|u(i)\|)$. Define $x' = (x'(i))$, $y' = (y'(i))$, $z' = (z'(i))$ by

$$x'(i) = \|u(i)\|\, x,\ y'(i) = \|u(i)\|\, y \text{ and } z'(i) = \|u(i)\|\, z.$$

Then

$$\rho_M(x') = \sum_{i=1}^{\infty} M(\|u(i)\|\,\|x\|) = \sum_{i=1}^{\infty} M(\|u(i)\|) = 1 < \infty,$$

and in the same way we get $\rho_M(y') = \rho_M(z') = 1$. Thus $x', y', z' \in S(l_M(X))$. Moreover, we see that $2x' = y' + z'$, $y' \neq z'$, so that x' is not an extreme point of $B(l_M(X))$, contradicting the rotundity of $l_M(X)$.

If (3) does not hold, then M is affine on some interval $[a,b] \subseteq [0, M^{-1}(1/2)]$. Since $b \leq M^{-1}(1/2)$. So $2M(b) \leq 1$, thus we can find $c \in (a,b)$ and $d > 0$ such that $2M(c) + M(d) = 1$.

Choose $x' \in S(X)$ and put

$$x = (cx', cx', dx', 0,0,0,\ldots),$$

then

$$\rho_M(x) = \sum_{i=1}^{\infty} M(\| x(i) \|)$$

$$= M(\| cx' \|) + M(\| cx' \|) + M(\| dx' \|)$$

$$= M(c \| x' \|) + M(c \| x' \|) + M(d \| x' \|)$$

$$= M(c) + M(c) + M(d) = 1,$$

hence $x \in S(l_M(X))$.

But $\|x(1)\|, \|x(2)\| \in \mathbb{R} \backslash S_M$, so it follows from Theorem 2.3 (3) that $x \notin \operatorname{Ext} B(l_M(X))$, which contradicts with rotundity of $l_M(X)$.

Sufficiency. Let $x \in S(l_M(X))$. It is obvious that (2) of Theorem 2.3 holds true by (2). Since $M \in \delta_2$ we have by Lemma 2.1 that $\rho_M(x) = 1$, so (1) of Theorem 2.3 is true.

Next, Let $I = \{i \in \mathbb{N} : \|x(i)\| \in \mathbb{R} \backslash S_M\}$. By (3), for any $i \in I$, we have $M(\| x(i) \|) > 1/2$. But since

$$\rho_M(x)=\sum_{i=1}^{\infty} M(\| x(i)\|)=1,$$

it implies that I contain atmost a single point. Hence, (3) of Theorem 2.3 is true. So we obtain by Theorem 2.3 that $x\in \operatorname{Ext} B(l_M(X))$.

Example 2.5. Theorem 2.4 may be false if X is not (R). Consider $X=\mathbb{R}^2$ equipped with the norm defined by

$$\|(\alpha_1,\alpha_2)\|=\sum_{j=1}^{2} |\alpha_j|,$$

we have that $(\mathbb{R}^2, \|\cdot\|)$ is a Banach space but not a rotund space. Let

$$x=\left(\left(\frac{1}{2},0\right), \left(0,\frac{1}{2}\right), (0,0), (0,0),\ldots\right),$$

$$y=((1,0),\ (0,0),\ (0,0),\ (0,0),\ldots.),$$

$$z=((0,0),\ (0,1),\ (0,0),\ (0,0),\ldots.).$$

Next, we define an Orlicz function $M:\mathbb{R}\to\mathbb{R}_+$ by,

$$M(u)=\begin{cases} 2u^2 & ;for\ u\in\left[-\frac{1}{2},\frac{1}{2}\right] \\ |u| & ;otherwise. \end{cases}$$

It is easy to see that $M\in\delta_2$ and M is strictly convex on $\left[0,M^{-1}\left(\frac{1}{2}\right)\right]$, and

$$\rho_M(x)=\sum_{i=1}^{\infty} M(\| x(i)\|)=M(\| x(1)\|)+M(\| x(2)\|)+M(\| x(3)\|)+\ldots.$$

$$=M\left(\frac{1}{2}\right)+M\left(\frac{1}{2}\right)+M(0)+M(0)+\ldots.$$

$$=\frac{1}{2}+\frac{1}{2}=1.$$

Similarly we can show that $\rho_M(y)=\rho_M(z)=1$, thus $x,y,z\in S(l_M(X))$. It is easy to see that $2x = y + z$, and $y \neq z$. This implies that x is not extreme point of $B(l_M(X))$. Hence, $l_M(X)$ is not a rotund space.

From Theorem 2.4, if $X=\mathbb{R}$ we have the following results.

Corollary 2.6. (Chen [1], Theorem 2.7) l_M is rotund iff

1. $M\in\delta_2$ and
2. M is strictly convex on $[0,M^{-1}(1/2)]$.

CHAPTER-IV

MUSIELAK-ORLICZ SEQUENCE SPACES

1. Preliminaries and Introduction

For any subset A of X by conv(A) ($\overline{\text{conv}}(A)$) we denote the *convex hull* (*closed convex hull*) of A.

A Banach space X is said to be *strictly convex* if for any $x,y \in S(X)$ with $\| x+y \| = 2$ we have $x = y$ (see [7]).

A Banach space X is said to have the *local uniform convexity property* (*local weak uniform convexity property*) if for every sequence $(x_n) \subset S(X)$ and $x \in S(X)$ with $\lim_{n\to\infty} \| x_n + x \| = 2$ we have $x_n \to x (x \xrightarrow{w} x)$ as $n \to \infty$ (see[7]).

The norm $\|\cdot\|$ is called *uniformly convex* (abbreviated as (UC)) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x,y \in S(X)$ the inequality $\|x-y\| > \varepsilon$ implies that

$$\left\| \frac{1}{2}(x+y) \right\| < 1-\delta. \qquad (1.1)$$

In this definition we can replace condition (1.1) by

$$\inf\{\| z \| : z \in \text{conv}(\{x,y\})\} < 1-\delta \quad \text{(see[40])}.$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \text{conv}(\{x\} \cup B(X)).$$

Rolewicz in [39], motivated the drop theorem of Danes [6], introduced the notion of the drop property for Banach spaces.

A Banach space X has the *drop property* (abbreviated as (D)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A Banach space X is said to have the *Kadec-Klee property* (or *property* (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

In [39] Rolewicz proved that if the Banach space X has the drop property, then X is reflexive. Montesinos [34] extended this result by showing that X has the drop property if and only if X is reflexive and X has the property (H).

A sequence $(x_n) \subset X$ is said to be an *ε-separate sequence* for some $\varepsilon > 0$ if

$$\operatorname{sep}(x_n) = \inf\{\| x_n - x_m \| : n \neq m\} > \varepsilon.$$

A Banach space X is said to have the *uniform Kadec-Klee property* (abbreviated as (UKK)) if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that if x is a weak limit of norm one ε-separate sequence, then

$$\| x \| < 1 - \delta.$$

A Banach space X is said to be *nearly uniformly convex* (abbreviated as (NUC)) (see [19]) if for every $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B$ with $\operatorname{sep}(x_n) > \varepsilon$, we have

$$\operatorname{conv}(x_n) \cap (1-\delta)B \neq \phi.$$

It is easy to see that every (NUC) space has the (UKK) property, and every Banach space with the (UKK) property has the property (H). Huff [19], proved that X is (NUC) if and only if X is reflexive and X has the (UKK) property.

For any subset C of X, the *Kuratowski measure* of C is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less then ε.

Goebel and Sekowski [12], extend the definition of uniform convexity replacing condition (1.1) by a condition involving the Kuratowski measure of noncompactness.

A norm $\|\cdot\|$ in a Banach space X is Δ-*uniformly convex* (abbreviated as ΔUC) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the closed unit ball $B(X)$ such that $\alpha(E) > \varepsilon$, we have

$$\inf\{\|x\| : x \in E\} < 1-\delta.$$

A Banach space X is said to have the *property* (β) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\alpha\big(D(x, B(X)) \setminus B(X)\big) < \varepsilon$$

whenever $1 < \| x \| < 1+\delta$.

The following equivalent form of the property (β) is very helpful in our work (see [28]).

A Banach space X has the property (β) if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\mathrm{sep}(x_n) \geq \varepsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

Rolewicz [40] showed that the property (β) follows from uniform convexity and that the property (β) implies the space is (ΔUC).

Resuming the above discussion we have

$$(\mathrm{UC}) \Rightarrow (\beta) \Rightarrow (\Delta\mathrm{UC}) \Leftrightarrow (\mathrm{NUC}) \Rightarrow (\mathrm{R}fx) \tag{1.2}$$

where (R*fx*) denotes the property of reflexivity.

A Banach space X is said to have the *local property* (β) if for every $\varepsilon > 0$ and $x \in S(X)$ there exists $\delta > 0$ such that for each sequence (x_n) in $B(X)$ with $\mathrm{sep}(x_n) > \varepsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

It is easy to see that a Banach space X with the local uniform convexity property has the local property (β) and a space X with the local property (β)has the Kadec-Klee property (see [29]).

We say that a Musielak-Orlicz function $\mathcal{M}$ satisfies the (*)-*condition* if for any $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that $M_i((1+\delta)u) \leq 1$ whenever $M_i(u) \leq 1-\varepsilon$ for all $i \in \mathbb{N}$ and $u \in \mathbb{N}$ (see [20]).

For more details we refer to [1] or [35].

In this chapter, criteria for the property (β), local uniform convexity, local weak uniform convexity and local property (β) in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm have been obtained.

2. Main Results

In order establish our new results, we need to recall some known facts.

Lemma 2.1. (see[20]) If a Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the (*)-condition and $\mathcal{M} \in \delta_2$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $||x|| < 1-\delta$, whenever $I_{\mathcal{M}}(x) < 1-\varepsilon$.

Lemma 2.2. (see[20]) If a Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the (*)- condition and $\mathcal{M} \in \delta_2$, then for each $\varepsilon > 0$ and each $c > 0$ there exists a $\delta > 0$ such that

$$|I_{\mathcal{M}}(x+y) - I_{\mathcal{M}}(x)| < \varepsilon$$

whenever $I_{\mathcal{M}}(x) \leq c$ and $I_{\mathcal{M}}(y) < \delta$.

Lemma 2.3. (see[16]) If a Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the $(*)$-condition and $\mathcal{M} \in \delta_2$, then $\|x\| = 1$ if and only if $I_{\mathcal{M}}(x) = 1$.

Lemma 2.4. (see [16]) If a Musielak-Orlicz function $\mathcal{M} \in \delta_2^*$, then there exists $\theta \in (0,1)$ and a sequence (h_i) in $\mathbb{R}_+$ with $\sum_{i=1}^{\infty} M_i(h_i) < \infty$ such that

$$M_i\left(\frac{u}{2}\right) \leq \frac{1-\theta}{2} M_i(u)$$

hold for every $i \in \mathbb{N}$ and u satisfying $M_i(h_i) \leq M_i(u) \leq 1$.

Lemma 2.5. If $\mathcal{M} \notin \delta_2^*$, then there exist $0 = I_0 < I_1 < I_2 < \ldots$ and $\{u_i^k\}$, for each $k \in \mathbb{N}$ and $i = I_{k-1}+1, \ldots, I_k$ such that

$$M_i(u_i^k) \leq \frac{1}{k}, \quad M_i\left(\frac{u_i^k}{2}\right) > \left(1 - \frac{1}{k}\right) \frac{M_i(u_i^k)}{2},$$

and

$$\sum_{i=I_{k-1}+1}^{I_k} M_i(u_i^k) > 1,$$

for $k = 1,2,\ldots$ (see[44]).

Theorem 2.6. If the Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the $(*)$-condition then the following statements are equivalent:

1. $l_{\mathcal{M}}$ has the property (β),
2. $l_{\mathcal{M}}$ is (ΔUC),

3. $l_{\mathcal{M}}$ has the drop property,

4. $\mathcal{M}$ satisfies the both δ_2-condition and δ_2^*-condition, that is, $l_{\mathcal{M}}$ is reflexive.

Proof. By (1.2), we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). To finish the proof, we have to show that (4) $\Rightarrow$ (1). Suppose that this implication is not true. Let $\varepsilon > 0$ be given. For each sequence $(x_n) \subset B(l_{\mathcal{M}})$ with $\mathrm{sep}(x_n) > \varepsilon$, we have $\mathrm{sep}\{(\sum_{i=m}^{\infty} x_n(i)e_i)\} > \varepsilon$ for all $m \in \mathbb{N}$. Hence for each $m \in \mathbb{N}$, there exists a $n_m \in \mathbb{N}$ such that

$$\left\| \sum_{i=m}^{\infty} x_n(i)e_i \right\| > \frac{\varepsilon}{2}$$

when $n > n_m$. Since $\mathcal{M} \in \delta_2$, there exists a $\delta > 0$ such that

$$I_{\mathcal{M}}\left(\sum_{i=m}^{\infty} x_n(i)e_i \right) > \delta, \quad m = 1,2,....$$

for all $n \in \mathbb{N}$. As $\mathcal{M} \in \delta_2^*$, there exists $\theta \in (0,1)$ and a sequence (h_i) in $\mathbb{R}_+$ with $\sum_{i=1}^{\infty} M_i(h_i) < \infty$ such that

$$M_i\left(\frac{u}{2}\right) \le \frac{1-\theta}{2} M_i(u)$$

holds for every $i \in \mathbb{N}$ and u satisfying $M_i(h_i) \le M_i(u) \le 1$. Using $\mathcal{M} \in \delta_2$ again, there exists $\delta_1 > 0$ such that

$$|I_{\mathcal{M}}(x+y)-I_{\mathcal{M}}(x)| < \frac{\delta\theta}{8}$$

whenever $I_{\mathcal{M}}(x) \leq 1$ and $I_{\mathcal{M}}(y) < \delta_1$. For any $x \in B(l_{\mathcal{M}})$, there exist $i_0 \in \mathbb{N}$ such that $\sum_{i=i_0+1}^{\infty} M_i(x(i)) < \delta_1$ and $\sum_{i=i_0+1}^{\infty} M_i(h_i) < \delta\theta/8$.

Hence

$$I_{\mathcal{M}}\left(\frac{x+x_n}{2}\right) = \sum_{i=1}^{\infty} M_i\left(\frac{x(i)+x_n(i)}{2}\right)$$

$$= \sum_{i=1}^{i_0} M_i\left(\frac{x(i)+x_n(i)}{2}\right) + \sum_{i=i_0+1}^{\infty} M_i\left(\frac{x(i)+x_n(i)}{2}\right)$$

$$\leq \frac{1}{2}\left(\sum_{i=1}^{i_0} M_i(x(i)) + \sum_{i=1}^{i_0} M_i(x_n(i))\right) + \sum_{i=i_0+1}^{\infty} M_i\left(\frac{x_n(i)}{2}\right) + \frac{\delta\theta}{8}$$

$$\leq \frac{1}{2}\left(\sum_{i=1}^{i_0} M_i(x(i)) + \sum_{i=1}^{i_0} M_i(x_n(i))\right) + \frac{1-\theta}{2} \sum_{i=i_0+1}^{\infty} M_i(x_n(i))$$

$$+ \sum_{i=i_0+1}^{\infty} M_i(h_i) + \frac{\delta\theta}{8}$$

$$\leq \frac{1}{2}\sum_{i=1}^{i_0} M_i(x(i)) + \frac{1}{2}\sum_{i=1}^{\infty} M_i(x_n(i)) - \frac{\theta}{2} \sum_{i=i_0+1}^{\infty} M_i(x_n(i)) + \frac{\delta\theta}{4}$$

$$\leq 1 - \frac{\theta}{2}\delta + \frac{\delta\theta}{4} = 1 - \frac{\delta\theta}{4},$$

when $n > n_{i_0}$.

Since $\mathcal{M} \in \bar{\delta}_2$ and satisfies the $(*)$-condition, by Lemma 2.1 there is a $0 < \Theta < 1$ such that

$$\left\| \frac{x + x_n}{2} \right\| < \Theta$$

when $n > n_{i_0}$, $l_{\mathcal{M}}$ has property (β).

Theorem 2.7. If the Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the $(*)$-condition then the following statements are equivalent:

1. $l_{\mathcal{M}}$ has the local uniform convexity property,
2. $l_{\mathcal{M}}$ has the local weak uniform convexity property,
3. (i) $\mathcal{M} \in \delta_2$ and $M_i \in SC[0, M_i^{-1}\left(\frac{1}{2}\right)]$,

   (ii) $\mathcal{M} \in \delta_2^*$ or $M_i \in SC[0, M_i^{-1}(1)]$ for all $i \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3). Since $l_{\mathcal{M}}$ has the local weak uniform convexity property, $l_{\mathcal{M}}$ is strictly convex. So (i) of condition (3) holds (see[21]). If (ii) of condition (3) is not true, then we may assume that there exists an affine interval $[a,b] \subset [M_1^{-1}\left(\frac{1}{2}\right), M_1^{-1}(1)]$ and $\mathcal{M} \notin \delta_2^*$. Then by lemma 2.5 there exist $0 = I_0 < I_1 < I_2 < \ldots$ and $\{u_i^k\}$ for $I = I_{k-1}+1, \ldots, I_k$ such that

$$M_i(u_i^k) \leq \frac{1}{k}, \quad M_i\left(\frac{u_i^k}{2}\right) > \left(1 - \frac{1}{k}\right)\frac{M_i(u_i^k)}{2}$$

and

$$\sum_{i=I_{k-1}+1}^{I_k} M_i(u_i^k) > 1,$$

for $k \in \mathbb{N}$. Take $t \geq 0$ such that $M_1(b) + M_2(t) = 1$. Choose natural numbers $m_k \in [I_{k-1}+1, I_k]$ such that

$$M_1(a) + M_2(t) + \sum_{i=I_{k-1}+1}^{m_k} M_i(u_i^k) \leq 1$$

and

$$M_1(a) + M_2(t) + \sum_{i=I_{k-1}+1}^{m_k+1} M_i(u_i^k) > 1.$$

Put $x = (b,t,0,0\ldots)$ and $x_n = (a,t,0,\ldots 0, \overset{(I_{n-1}+3)th}{u_{I_{n-1}+1}^n}, \ldots\ldots\ldots, \overset{(m_n+2)th}{u_{m_n}^n}, 0,0\ldots\ldots)$ for $n = 1,2,\ldots$. It is clear that $\|x\| = 1$ and $\lim_{n\to\infty} \|x_n\| = 1$. Moreover, we have

$$I_{\mathcal{M}}\left(\frac{x_n + x}{2}\right) = M_1\left(\frac{a+b}{2}\right) + M_2(t) + \sum_{i=I_{n-1}+1}^{m_n} M_i\left(\frac{u_i^n}{2}\right)$$

$$\geq \frac{1}{2}(M_1(a) + M_1(b)) + M_2(t) + \frac{1-\frac{1}{n}}{2} \sum_{i=I_{n-1}+1}^{m_n} M_i(u_i^n)$$

$$= \frac{1}{2} + \frac{1}{2}\left(M_1(a) + M_2(t) + \sum_{i=I_{n-1}+1}^{m_n} M_i(u_i^n)\right) - \frac{1}{2n} \sum_{i=I_{n-1}+1}^{m_n} M_i(u_i^n) \to 1$$

Hence $\|x_n + x\| \to 2$ as $n \to \infty$. It is obvious that $x_n \xrightarrow{w} x$ is impossible, a contradiction.

Sufficiency, we divide the proof into two parts:

Part (I). $\mathcal{M}\in\delta_2$, $\mathcal{M}\in\delta_2^*$ and $M_i\in SC[0,M_i^{-1}\left(\frac{1}{2}\right)]$.

For every sequence $(x_n)\subset S(l_{\mathcal{M}})$ and $x\in S(l_{\mathcal{M}})$ with $\|x_n+x\|\to 2$, we have (x_n) is compact in $S(l_{\mathcal{M}})$ thanks to Theorem 2.6. Hence there exists a subsequence $(x_{n_t})\subset(x_n)$ and $x'\in S(l_{\mathcal{M}})$ such that $x_{n_t}\to x'$. Furthermore, we have $\|x+x'\|=2$. As $l_{\mathcal{M}}$ is strictly convex, we get $x'=x$. Thus we obtain $x_n\to x$.

Part II. $\mathcal{M}\in\delta_2$ and $M_i\in SC[0,M_i^{-1}(1)]$.

First, we will prove $x_n(i)\to x(i)$ for $i=1,2,\ldots$ If not, we may assume that, without loss of generality, there exist a $i_0\in\mathbb{N}$ and $\varepsilon_0>0$ such that $|x_n(i_0)-x(i_0)|\geq\varepsilon_0$ for $n\in\mathbb{N}$. Since $M_{i_0}\in SC[0,M_{i_0}^{-1}(1)]$, there exists a $\theta\in(0,1)$ such that

$$M_{i_0}\left(\frac{x_n(i)+x(i)}{2}\right)<\frac{1-\theta}{2}\left(M_{i_0}(x_n(i))+M_{i_0}(x(i))\right)$$

(see[1]). Hence

$$1\leftarrow I_{\mathcal{M}}\left(\frac{x_n+x}{2}\right)=\sum_{i=1,i\neq i_0}^{\infty}M_i\left(\frac{x_n(i)+x(i)}{2}\right)+M_{i_0}\left(\frac{x_n(i_0)+x(i_0)}{2}\right)$$

$$\leq\frac{1}{2}\sum_{i=1,i\neq i_0}^{\infty}\left(M_i(x_n(i))+M_i(x(i))\right)+\frac{1-\theta}{2}\left(M_{i_0}(x_n(i_0))+M_{i_0}(x(i_0))\right)$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} \left(M_i(x_n(i)) + M_i(x(i))\right) - \frac{\theta}{2} M_{i_0}(|x_n(i_0) - x(i_0)|)$$

$$\leq 1 - \frac{\theta}{2} M_{i_0}(\varepsilon_0),$$

a contradiction.

Last, we will prove (x_n) has equi-absolutely continuous norm. For any $\varepsilon > 0$ there exists a $i_1 \in \mathbb{N}$ such that $\sum_{i=i_1+1}^{\infty} M_i(x(i)) < \varepsilon$, i.e., $\sum_{i=1}^{i_1} M_i(x(i)) > 1 - \varepsilon$. Using $x_n(i) \to x(i)$, there exists a $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{i_1} M_i(x_n(i)) > 1 - \varepsilon$ when $n > n_0$. Hence

$$\sum_{i=i_1+1}^{\infty} M_i(x_n(i)) = 1 - \sum_{i=1}^{i_1} M_i(x_n(i)) < 1 - (1 - \varepsilon) = \varepsilon.$$

Since $\mathcal{M} \in \delta_2$, we have that (x_n) has equi-absolutely continuous norm.

So, $x_n \to x$.

Observing the proof of Theorem 2.7 and using Theorem 2.6 and the fact that if the Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the (*)-condition then $l_{\mathcal{M}}$ has the Kadec-Klee property if and only if (1) $\mathcal{M} \in \delta_2$ (see [20]), we can easily obtain the following result.

Theorem 2.8. If Musielak-Orlicz function $\mathcal{M} = (M_n)$ satisfies the (*)-condition then $l_{\mathcal{M}}$ has the local property (β) if and only if (1) $\mathcal{M} \in \delta_2$ and (2) $\mathcal{M} \in \delta_2^*$ or $M_i \in SC[0, M_i^{-1}(1)]$.

Recall that the *Nakano space* $l^{(p_i)}$ is the Musielak-Orlicz sequence space with $\mathcal{M} = (M_i)$ where

$$M_i(u) = |u|^{p_i}, \qquad 1 < p_i < +\infty, \quad i \in \mathbb{N}$$

Corollary 2.9. $l^{(p_i)}$ has the property (β) if and only if

$$1 < \liminf_{i\to\infty} p_i \le \limsup_{i\to\infty} p_i < +\infty .$$

Proof. If $M_i(u) = |u|^{p_i}$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$, then the complementary function $\mathcal{N}_i$ of M_i is defined by the formula

$$\mathcal{N}_i(u) = c_i\, |u|^{q_i},$$

where $1/p_i + 1/q_i = 1$ and $c_i = (p_i)^{1/p_i}(q_i)^{1/q_i}$ for all $i \in \mathbb{N}$. It is easy to see that $\mathcal{M} \in \delta_2$ if and only if $\limsup_{i\to\infty} p_i < +\infty$. Moreover, $\mathcal{N} \in \delta_2^*$ if and only if $1 < \liminf_{i\to\infty} p_i$.

Corollary 2.10. The following statements are equivalent:

1. $l^{(p_i)}$ has the local uniform strict convexity property,
2. $l^{(p_i)}$ has the local property (β),
3. $1 < \liminf_{i\to\infty} p_i$.

CHAPTER-V

SEQUENCES DEFINED BY MUSIELAK-ORLICZ FUNCTIONS

1. Preliminaries and Introduction

The spaces of strongly summable sequences were discussed by Maddox [31]. Parashar and Choudhary [37] defined these spaces by using the idea of Orlicz function as follows:

Let $p = (p_k)$ be a sequence of positive real numbers and ω be the space of all real sequences. Then

$$W_0(M,p) = \left\{ x \in \omega : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|x_k|}{\lambda}\right)\right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \lambda > 0 \right\},$$

$$W(M,p) = \{x \in \omega : x - le \in W_0(M,p),\ l > 0\},$$

$$W_\infty(M,p) = \left\{ x \in \omega : \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|x_k|}{\lambda}\right)\right]^{p_k} < \infty, \text{ for some } \lambda > 0 \right\}.$$

If $M(x) = x$, then the above spaces are deduced to $[C,1,p]_0$, $[C,1,p]$ and $[C,1,p]_\infty$ respectively. For $p_k = p > 0$ for each k, we denote these sequence spaces by $W_0^p(M)$, $W^p(M)$ and $W_\infty^p(M)$ respectively.

Let X be a Banach space and $\omega(X)$ denote the space of all sequences $x = (x_k)$ in X. A matrix $A = (a_{nk})_{n,k}^{\infty}$ is called *regular* on $\omega(X)$

if A maps $c(X)$ into $c(X)$ and $\lim_n A_n(x) = \lim_k x_k$ in X. It is known that a matrix A is regular on $\omega(X)$ if and only if it is regular on ω. The necessary and sufficient conditions for A to be regular [14] on ω are (i) $\sup_n \sum_k |a_{nk}| < \infty$, (ii) $\lim_n a_{nk} = 0$ for each k and (iii) $\lim_n \sum_k a_{nk} = 1$.

These are well-known Silverman-Toeplitz conditions (see[14]). A matrix A is said to be *uniformly regular* if it regular, $a_{nk} \geq 0$ and $\limsup_n \sum_k |a_{nk}| = 0$.

We define the following sequence spaces

Let $A = (a_{nk})_{n,k}^{\infty}$ be a non-negative regular matrix and $\mathcal{M} = (M_k)$ a sequence of Orlicz functions such that each M_k satisfies δ_λ-condition. Then for $p > 0$.

$$W_0^p(\mathcal{M},A,X) = \left\{ x \in \omega(X) : \lim_n \sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\lambda} \right) \right)^p = 0 \text{ , for some } \lambda > 0 \right\},$$

$W^p(\mathcal{M},A,X) = \{x \in \omega$: there exists $x_0 \in X$, $(x_k - x_0) \in W_0^p(\mathcal{M},A,X)\}$.

For $x \in W^p(\mathcal{M},A,X)$, we write $x_k \to x_0$ $(W^p(\mathcal{M},A,X))$.

If $M_k(x) = x$ for each k, then these spaces are reduced to $W_0^p(A,X)$ and $W^p(A,X)$ respectively, where

$$W^p(A,X) = \left\{ x \in \omega(X) : \lim_n \sum_k a_{nk} \| x_k - x_0 \|^p = 0 \right\}.$$

If (M_k) is replaced by (f_k) a sequence of modulus functions, then the above spaces are reduced to the spaces defined by Kolk [26].

In this chapter we define the notion of strong summability by a sequence of Orlicz functions and consider some inclusion relation (see [22]).

2. Inclusion Relations

In this section, we prove the following results.

Theorem 2.1. W_0^p (A,X)$\subset$ $W_0^p(\mathcal{M},A,X)$ if and only if

$$\lim_{t\to 0^+} \sup_k M_k(t) = 0 \qquad (t>0). \tag{2.1.1}$$

Proof. Let W_0^p(A,X)$\subset W_0^p(\mathcal{M},A,X)$. If we take $A=I$ (unit matrix), then this inclusion is reduced to

$$c_0(X) \subset c_0(\mathcal{M},X)$$

where $c_0(\mathcal{M},X) = \left\{ x \in \omega(X) : \lim_n \sum_k \left(M_k\left(\frac{\| x_k \|}{\lambda}\right)\right) = 0, \text{ for some } \lambda > 0 \right\}$

Suppose that (2.1.1) fails to hold. Then there exists a number $\varepsilon_0 > 0$ and an index sequence (k_i) such that

$$M_{k_i}(t) \geq \varepsilon_0 \qquad (i=1,2..) \tag{2.1.2}$$

for a positive sequence $(t_i)\in c_0$. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_i y, \; for \;\; k = k \text{ and for a fixed } y\in X \text{ with } \|y\| = 1; \\ 0, \;\; if \;\; k \neq k_i. \end{cases}$$

Then $x \in c_0(X)$ since $t_i \in c_0$, and hence $x\in c_0(\mathcal{M},X)$. On the other hand, by (2.1.2) and δ_λ-condition we have

$$M_{k_i}\left(\frac{x_{k_i}}{\lambda}\right)-M_{k_i}\left(\frac{t_i}{\lambda}\right)\geq\frac{\varepsilon_0}{K\lambda},\ K>0,\ \lambda>1,\ (i=1,2....)$$

i.e., $x\notin c_0(m,X)$, a contradiction. Therefore (2.1.1) must hold.

Conversely, suppose that (2.1.1) holds. Then for every $\varepsilon>0$ there exists a number δ such that $0<\delta<1$ and

$$M_k(t)<\frac{\varepsilon^{\frac{1}{p}}}{\|A\|},\ k=1,2....\ \text{for}\ t\leq\delta \tag{2.1.3}$$

For a sequence $x=(x_k)\in W_0^p(A,X)$ let

$$T_n=\sum_k a_{nk}\left(\frac{\|x_k\|}{\lambda}\right)^p,$$

so that $\lim_n T_n=0$. Now

$$\sum_k a_{nk}\left[M_k\left(\frac{\|x_k\|}{\lambda}\right)\right]^p=\Sigma'+\Sigma'' \tag{2.1.4}$$

where Σ' is the sum over k such that $\frac{\|x_k\|}{\lambda}\leq\delta$ and Σ'' is the sum over k such that $\frac{\|x_k\|}{\lambda}>\delta$.

Since A is regular and by (2.1.3), we have

$$\Sigma'<\varepsilon \tag{2.1.5}$$

By (2.1.1), we have

$$\sup_k M_k(\delta)=H<\infty\quad\text{for}\quad\frac{\|x_k\|}{\lambda}>\delta>0. \tag{2.1.6}$$

Since each M_k is non-decreasing and convex, we have by (2.1.6) and δ_λ-condition that for $K > 0$

$$M_k\left(\frac{\| x_k \|}{\lambda}\right) = M_k\left(\delta\delta^{-1}\frac{\| x_k \|}{\lambda}\right),$$

$$\leq K\delta^{-1}\frac{\| x_k \|}{\lambda}M_k(\delta),$$

$$< K\delta^{-1}H\frac{\| x_k \|}{\lambda},$$

i.e. $$\Sigma'' < (K\delta^{-1}H)^p T_n. \tag{2.1.7}$$

Hence $\Sigma'' \to 0$ as $n\to\infty$. Therefore $x \in W_0^p(\mathcal{M},A,X)$. This completes the proof of the theorem.

Theorem 2.2. $W_0^p(\mathcal{M},A,X) \subset W_0^p(A,X)$ if and only if

$$\inf_k M_k(t) > 0 \qquad (t > 0) \tag{2.2.1}$$

Proof. Let $W_0^p(\mathcal{M},A,X) \subset W_0^p(A,X)$. Suppose that (2.2.1) does not hold. Then

$$\inf_k M_k(t) = 0 \quad (t > 0) \tag{2.2.2}$$

and thus we can choose an index sequence (k_i) such that

$$M_{k_i}(t_i) < \frac{1}{i} \qquad (i = 1,2,....). \tag{2.2.3}$$

Now, define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0 y, & \textit{for } k = k_i \textit{ where } y \in X \textit{ with } \| y \| = 1 \textit{ and } t_0 > 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\| x_k \| = \| x_{k_i} \| = t_0$, and so by (2.2.2) and (2.2.3) we get

$$\lim_k M_k\left(\frac{\| x_k \|}{\lambda}\right) = 0,$$

and hence

$$\lim_k \left[M_k\left(\frac{\| x_k \|}{\lambda}\right)\right]^p = 0.$$

Further by regularity of A, we have

$$\lim_k \sum_k a_{nk} \left[M_k\left(\frac{\| x_k \|}{\lambda}\right)\right]^p = 0,$$

i.e. $x = (x_k) \in W_0^p(\mathcal{M}, A, X)$. But on the other hand

$$\lim_n \sum_k a_{nk} \| x_k \|^p = t_0^p \lim_n \sum_k a_{nk} = t_0^p, \text{ since } A \text{ is regular.}$$

Hence $x \notin W_0^p(A, X)$. Which contradicts that $W_0^p(\mathcal{M}, A, X) \subset W_0^p(A, X)$.

Hence (2.2.1) must hold.

Conversely, let (2.2.1) hold and $x \in W_0^p(\mathcal{M}, A, X)$. Suppose that $x \notin W_0^p(\mathcal{M}, A, X)$. Then for some number $\varepsilon_0 > 0$ and index k_0 we have $\| x_{k_i} \| > \varepsilon_0$ $(i \in \mathbf{N})$ for some subsequence of indices (k_i), since A is regular. Thus

$$M_k\left(\frac{\varepsilon_0}{\lambda}\right) < M_k\left(\frac{\| x_{k_i} \|}{\lambda}\right) \text{ for some } \lambda > 0,$$

and further by regularity of A, we have $\lim_k M_k\left(\frac{\varepsilon_0}{\lambda}\right) = 0$ which contradicts (2.2.1). Hence $x \in W_0^p(A, X)$. This completes the proof of the theorem.

3. A-Statistical Convergence

In this section we find relation of A-statistical convergence with strong A-summability defined by a sequence $\mathcal{M} = (M_k)$ of Orlicz functions.

Let $K = \{k_i\}$ be an index set, i.e. precisely the sequence (k_i) of indices. Let ϕ^k be the characteristic sequence of K, i.e. $\phi^k = (\phi_j^k)$, where

$$\phi_j^k = \begin{cases} 1, & \text{If } j = k,\ i = 1,2,\ldots. \\ 0, & \text{otherwise.} \end{cases}$$

If ϕ^k is $(C,1)$-summable then the limit

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \phi_j^k$$

is called the asymptotic density of K and is denoted by $\delta(K)$.

An index set $K = \{k_i\}$ is said to have A-density if

$$\delta_A(K) = \lim_n A_n \phi^k = \lim_n \sum_{k \in K} a_{nk}$$

exists, where $A = (a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix (cf[25]).

The idea of statistical convergence was introduced by Fast [9] and studied by various authors, e.g. by Salat [42], Freedman and Sember [10], Fridy [11], Connor [5], and Kolk [25].

A sequence $x = (x_k) \in \omega(X)$ is said to be A-statistically convergent to x_0, (see [25]) i.e. $x_k \to x_0(S_A(X))$ if for every $\varepsilon > 0$, $\delta_A(L_\varepsilon) = 0$, where $L_\varepsilon = \{k : \| x_k - x_0 \| \geq \varepsilon\}$. We denote by $S_A(X)$ the set of all A-statistically convergent sequences in X. If A is C_l-matrix, then A-statistical convergence is reduced to the statistical convergence.

Example 3.1. Define $x_k = 1$ if k is a square and $x_k = 0$ otherwise. Then $|\{k \leq n : x_k \neq 0\}| \leq (n)^{1/2}$, so $x = (x_k)$ is statistically convergent to 0.

Theorem 3.2. Let A be uniformly regular matrix and the sequence $\mathcal{M} = (M_k)$ be a pointwise convergent. Then

$$x_k \to x_0(W_0^p(\mathcal{M}, A, X)) \Rightarrow x_k \to x_0(S_A(X)),$$

if and only if

$$\lim_k M_k(t) > 0 \quad (t > 0). \tag{3.1.1}$$

Proof. Let $\varepsilon > 0$. Then as in [27, Theorem 3.8], we can find numbers $s > 0$ and $r \in \mathbb{N}$ such that

$$\sum_{\substack{k \in L_\epsilon \\ k \geq r}} a_{nk} \leq s^{-p} \sum_{k \geq r} a_{nk} \left[M_k \left(\frac{\| x_k - x \|}{\lambda} \right) \right]^p. \tag{3.1.2}$$

Where $L_\varepsilon = \{k : \| x_k - x_0 \| \geq \varepsilon\}$. Since $x_k \to x_0(W_0^p(\mathcal{M}, A, X))$ implies that $\delta_A(L_\varepsilon) = \lim_n \sum_{k \in L_\varepsilon} a_{nk} = 0$. Therefore $x_k \to x_0(S_A(X))$.

Conversely, suppose that $x_k \to x_0(W_0^p(\mathcal{M}, A, X) \Rightarrow x_k \to x_0(S_A(X))$.

If (3.1.1) is not true we have

$$\lim_k M_k(t_0) = 0 \text{ for some } t_0 > 0.$$

Since A is uniformly regular, by Lemma 2.4 of Kolk [27], there exists an infinite index set $K = (k_i)$ with $\delta_A(K) = 0$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} 0, & k \in K, \\ t_0 z, & otherwise; \end{cases}$$

where $z \in X$ with $\|z\| = 1$. Then

$$\lim_k \left[M_k \left(\frac{\| y_k \|}{\lambda} \right) \right]^p = 0,$$

and by the regularity of A we have

$$y_k \to 0(W^p(\mathcal{M}, A, X)).$$

But for $0 < \varepsilon \leq t_0$,

$$\delta_A(k : \| y \| \geq \varepsilon) = \lim_n \sum_k a_{nk} - \delta_A(K) = 1 - 0 = 1.$$

Thus y_k does not$\to$ $0(S_A(X))$, i.e. contradiction to the hypothesis. Hence (3.1.1) must hold. This completes the proof of the theorem.

CHAPTER-VI

SOME NEW SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION IN A LOCALLY CONVEX SPACE

1. Preliminaries and Introduction

Kizmaz [24] introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\},$$

for $X = l_\infty, c, c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

Later on the notion was generalized by Et and Colak [8] as follows:

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

for $X = l_\infty, c, c_0$, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$ and $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ such that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

The sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, $c_0(\Delta^m)$ are Banach spaces, normed by

$$\| x \|_\Delta = \sum_{i=0}^{m} | x_i | + \| \Delta^m x \|_\infty .$$

A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $| \alpha_k | \le 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be *monotone* if E contains the canonical preimages of all its steps.

Let M be an Orlicz function, X be a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q and $p = (p_k)$ be a sequence of positive real numbers. The symbol $\omega(X)$ denotes the space of all sequences defined over X. We define the following sequence spaces:

$$W(\Delta^m, M, p, q) = \{x \in \omega(X) : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k - L}{\lambda}\right)\right)\right]^{p_k} \to 0,$$

as $n \to \infty$ for some $\lambda > 0$ and $L > 0\}$

$$W_0(\Delta^m, M, p, q) = \{x \in \omega(X) : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k}{\lambda}\right)\right)\right]^{p_k} \to 0,$$

as $n \to \infty$ for some $\lambda > 0\}$

$$W_\infty(\Delta^m, M, p, q) = \{x \in \omega(X) : \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k}{\lambda}\right)\right)\right]^{p_k} < \infty,$$

for some $\lambda > 0\}$

We denote $W(\Delta^m, M, p, q)$, $W_0(\Delta^m, M, p, q)$ and $W_\infty(\Delta^m, M, p, q)$ as $[C, \Delta^m, 1, q]$, $[C, \Delta^m, 1, q]_0$ and $[C, \Delta^m, 1, q]_\infty$ respectively for $p_k = 1$ for all $k \in \mathbb{N}$ and $M(x) = x$.

We get the following sequence spaces from the above sequence spaces on giving particular values to X, q and m.

1. If $X = \mathbb{C}$ and $q(x) = |x|$, then $W(\Delta^m, M, p, q) = W(M,p)(\Delta^m)$, $W_0(\Delta^m, M, p, q) = W_0(M,p)(\Delta^m)$ and $W_\infty(\Delta^m, M, p, q) = W_\infty(M,p)(\Delta^m)$.

Where $W_\infty(M,p)(\Delta^m) = \{x \in \omega : \sup\limits_n \frac{1}{n} \sum\limits_{k=1}^{n} \left[M\left(\left| \frac{\Delta^m x_k}{\lambda} \right| \right) \right]^{p_k} < \infty,$

for some $\lambda > 0\}$.

2. If $m = 0$, then $W(\Delta^m, M, p, q) = W(M, p, q)$, $W_0(\Delta^m, M, p, q) = W_0(M, p, q)$ and $W_\infty(\Delta^m, M, p, q) = W_\infty(M, p, q)$. Where

$$W_\infty(M,p,q) = \{x \in \omega(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M\left(q\left(\frac{x_k}{\lambda} \right) \right) \right]^{p_k} < \infty, \text{ for some } \lambda > 0\}.$$

3. If $m = 0$, $q(x) = |x|$ and $X = \mathbb{C}$ then $W(\Delta^m, M, p, q) = W(M,p)$, $W_0(\Delta^m, M, p, q) = W_0(M,p)$ and $W_\infty(\Delta^m, M, p, q) = W_\infty(M,p)$ (see Parashar and Choudhary [37]).

4. Taking $p_k = 1$ for all $k \in \mathbb{N}$, we denote the above defined sequence spaces as $W(\Delta^m, M, p, q) = W(\Delta^m, M, q)$, $W_0(\Delta^m, M, p, q) = W_0(\Delta^m, M, q)$ and $W_\infty(\Delta^m, M, p, q) = W_\infty(\Delta^m, M, q)$.

If $x \in W(\Delta^m, M, q)$, then we say that (x_k) is strongly Δ_q^m-Cesaro summable with respect to the Orlicz function M.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le \sup\limits_k p_k = G$, $D = \max(1, 2^{H-1})$.

Then for $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k+b_k|^{p_k} \le D\Big\{ |a_k|^{p_k} + |b_k|^{p_k} \Big\}, \tag{1.1}$$

(see for instance Maddox [32]).

In this chapter, we introduce some new sequence spaces with respect to an Orlicz function and for all $q \in Q$, where Q is the set of continuous seminorms.

2. Main Results

Theorem 2.1. Let the sequence (p_k) be bounded, then $W(\Delta^m, M, p, q)$, $W_0(\Delta^m, M, p, q)$ and $W_\infty(\Delta^m, M, p, q)$ are linear spaces.

Proof. Let $x, y \in W_0(\Delta^m, M, p, q)$ and $\alpha, \beta \in \mathbb{C}$, then there exists positive numbers λ_1, λ_2 such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k}{\lambda_1}\right)\right)\right]^{p_k} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k}{\lambda_2}\right)\right)\right]^{p_k} = 0.$$

Let $\lambda_3 = \max(2|\alpha|\lambda_1, 2|\beta|\lambda_2)$. Since M is non-decreasing and convex, q is a seminorm and Δ^m is linear,

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m(\alpha x_k+\beta y_k)}{\lambda_3}\right)\right)\right]^{p_k} = \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\alpha\Delta^m x_k+\beta\Delta^m y_k}{\lambda_3}\right)\right)\right]^{p_k}$$

$$\leq \frac{D}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m x_k}{\lambda_1}\right)\right)\right]^{p_k} + \frac{D}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^m y_k}{\lambda_2}\right)\right)\right]^{p_k}$$

$\to 0$, as $n \to \infty$.

This proves that $W_0(\Delta^m, M,p,q)$ is a linear space

The other cases are routine works in view of the above proof.

Theorem 2.2. The space $W(\Delta^m,M,p,q)$, $W_0(\Delta^m,M,p,q)$ and $W_\infty(\Delta^m,M,p,q)$ are paranormed space (not totally paranormed), paranormed by

$$g_\Delta(x) = \inf\left\{\lambda^{\frac{p_n}{H}} : \sup_{k\geq 1} M\left(q\left(\frac{\Delta^m x_k}{\lambda}\right)\right) \leq 1,\ \lambda > 0,\ n \in \mathbb{N}\right\},$$

where $H = \max(1, \sup p_k)$.

Proof. Consider $W_\infty(\Delta^m,M,p,q)$. Then clearly $g_\Delta(x) \geq 0$, $g_\Delta(x) = g_\Delta(-x)$ and $g_\Delta(\bar{\theta}) = 0$. Let $(x_k),(y_k) \in W_\infty(\Delta^m,M,p,q)$, then there exists $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$\sup_{k\geq 1} M\left(q\left(\frac{\Delta^m x_k}{\lambda_1}\right)\right) \leq 1$$

and

$$\sup_{k\geq 1} M\left(q\left(\frac{\Delta^m y_k}{\lambda_2}\right)\right) \leq 1.$$

Let $\lambda = \lambda_1 + \lambda_2$, then we have

$$\sup_{k\geq 1} M\left(q\left(\frac{\Delta^m(x_k+y_k)}{\lambda}\right)\right) \leq \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\sup_{k\geq 1} M\left(q\left(\frac{\Delta^m x_k}{\lambda_1}\right)\right)+\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)\sup_{k\geq 1} M\left(q\left(\frac{\Delta^m y_k}{\lambda_2}\right)\right) \leq 1$$

Thus we have $g_\Delta(x+y) \leq g_\Delta(x)+g_\Delta(y)$.

Finally let μ be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality

$$g_\Delta(\mu x) = \inf\left\{\lambda^{\frac{p_n}{H}} : M\left(q\left(\frac{\Delta^m \mu x_k}{\lambda}\right)\right) \leq 1,\ \ \lambda > 0, n \in \mathbb{N}\right\},$$

$$= \inf\left\{(r\,|\,\mu\,|)^{\frac{p_n}{H}} : M\left(q\left(\frac{\Delta^m x_k}{r}\right)\right) \leq 1,\ \ \lambda > 0, n \in \mathbb{N}\right\},$$

where $r = \dfrac{\lambda}{|\,\mu\,|}$.

This completes the proof of the theorem.

Theorem 2.3. *Let M, M_1, M_2 be Orlicz functions those satisfy δ_2-condition. Then*

1. $W_0(\Delta^m, M_1, p, q) \subseteq W_0(\Delta^m, M0M_1, p, q)$.
2. $W_0(\Delta^m, M_1, p, q) \cap W_0(\Delta^m, M_2, p, q) \subseteq W_0(\Delta^m, M_1+M_2, p, q)$.

Proof. (1) Let $(x_k) \in W_0(\Delta^m, M_1, p, q)$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_k = M_1\left(q\left(\dfrac{\Delta^m x_k}{\lambda}\right)\right)$ and consider

$$\sum_{k=1}^{n} [M(y_k)]^{p_k} = \sum\nolimits_1 [M(y_k)]^{p_k} + \sum\nolimits_2 [M(y_k)]^{p_k},$$

where the first summation is over $y_k \leq \delta$. Since M is continuous, we have

$$\sum\nolimits_1 [M(y_k)]^{p_k} < n\varepsilon^H \tag{2.3.1}$$

and for $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since M is non-decreasing and convex, if follows that

$$M(y_k) < M\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(2\frac{y_k}{\delta}\right).$$

Since M satisfies δ_2-condition, therefore there exists $K \geq 1$ such that

$$M(y_k) < \frac{1}{2}K\frac{y_k}{\delta}M(2) + \frac{Ky_k}{2\delta}M(2) = KM(2)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{n}\sum\nolimits_2 [M(y_k)]^{p_k} \leq \max\left(1, \left(KM(2)\delta^{-1}\right)^H\right)\frac{1}{n}\sum_{k=1}^{n} [y_k]^{p_k} \to 0, \; as \; n \to \infty \tag{2.3.2}$$

By (2.3.1) and (2.3.2) we have $(x_k) \in W_0(\Delta^m, M0M_1, p, q)$.

Let $(x_k) \in W_0(\Delta^m, M_1, p, q) \cap W_0(\Delta^m, M_2, p, q)$. Then using (1.1) it can be shown that $(x_k) \in W_0(\Delta^m, M_1+M_2, p, q)$. Hence

$W_0(\Delta^m, M_1, p, q) \cap W_0(\Delta^m, M_2, p, q) \subseteq W_0(\Delta^m, M_1+M_2, p, q)$.

The proof of the following result is a routine work in view of the above Theorem.

Corollary 2.4. Let M, M_1, M_2 be Orlicz functions those satisfy δ_2-condition. Then

1. $W(\Delta^m, M_1, p, q) \subseteq W(\Delta^m, M0M_1, p, q)$.

2. $W(\Delta^m, M_1, p, q) \cap W(\Delta^m, M_2, p, q) \subset W(\Delta^m, M_1+M_2, p, q)$.

3. $W_\infty(\Delta^m, M_1, p, q) \subseteq W_\infty(\Delta^m, M0M_1, p, q)$.

4. $W_\infty(\Delta^m, M_1, p, q) \cap W_\infty(\Delta^m, M_2, p, q) \subseteq W_\infty(\Delta^m, M_1+M_2, p, q)$.

The proof of the following result is a consequence of Theorem 2.3 (1), Corollary 2.4 (1) and Corollary 2.4 (2).

Proposition 2.5. Let M be an Orlicz function which satisfies the δ_2-condition. Then

1. $[C, \Delta^m, 1, q]_0 \subset W_0(\Delta^m, M, q)$.

2. $[C, \Delta^m, 1, q] \subset W(\Delta^m, M, q)$.

3. $[C, \Delta^m, 1, q]_\infty \subset W_\infty(\Delta^m, M, q)$.

The proof of the following result is a routine work.

Proposition 2.6. $W(\Delta^{m-1}, M, p, q) \subset W_0(\Delta^m, M, p, q)$.

Theorem 2.7. Let $m \geq 1$, then the following inclusions are strict.

1. $W_0(\Delta^{m-1}, M, p, q) \subset W_0(\Delta^m, M, p, q)$.

2. $W(\Delta^{m-1}, M, p, q) \subset W(\Delta^m, M, p, q)$.

3. $W_\infty(\Delta^{m-1}, M, p, q) \subset W_\infty(\Delta^m, M, p, q)$.

Proof. We prove the case (1) only. The other cases follow in a similar way. Let $x \in W_0(\Delta^{m-1}, M, p, q)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^{m-1}x_k}{\lambda}\right)\right)\right]^{p_k} \to 0, \text{ as } n\to\infty \text{ for some } \lambda > 0.$$

Since M is non-decreasing, convex function and q is a seminorm, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^{m}x_k}{2\lambda}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}}{2\lambda}\right)\right)\right]^{p_k}$$

$$\le D\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1}{2}M\left(q\left(\frac{\Delta^{m-1}x_k}{\lambda}\right)\right)\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{n}\left[\frac{1}{2}M\left(q\left(\frac{\Delta^{m-1}x_{k+1}}{\lambda}\right)\right)\right]^{p_k}\right\}$$

$$\le \frac{2D}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\Delta^{m-1}x_k}{\lambda}\right)\right)\right]^{p_k}$$

$\to 0$, as $n\to\infty$, by (2.3.2).

This completes the proof. In general $W_0(\Delta^i, M, p, q) \subset W_0(\Delta^m, M, p, q)$ for all $i = 1,2,3,\ldots, m-1$ and the inclusion is strict.

To show that the inclusion is strict, consider the following example.

Example 2.8. Let $X = \mathbb{C}$ $M(x) = x$, $q(x) = |x|$ and put $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence $(x_k) = (k^{m-1})$. Then $x \in W_0(\Delta^m, M, p, q)$ but $x \notin W_0(\Delta^{m-1}, M, p, q)$, since $\Delta^m x_k = 0$, $\Delta^{m-1} x_k = (-1)^{m-1}(m-1)!$ for all $k \in \mathbb{N}$.

Proposition 2.9. For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two seminorms q_1, q_2 we have

1. $W_0(\Delta^m, M, p, q_1) \cap W_0(\Delta^m, M, p, q_2) \neq \phi$.
2. $W(\Delta^m, M, p, q_1) \cap W(\Delta^m, M, p, q_2) \neq \phi$.
3. $W_\infty(\Delta^m, M, p, q_1) \cap W_\infty(\Delta^m, M, p, q_2) \neq \phi$.

Proof. Since the zero element belongs to each of the above classes of sequences, thus the intersection is nonempty

Theorem 2.10. Let $0 < p_k \leq r_k$ and $\left(\frac{r_k}{p_k}\right)$ be bounded, then

$$W(\Delta^m, M, r, q) \subset W(\Delta^m, M, p, q).$$

Proof. Let $(x_k) \in W(\Delta^m, M, r, q)$. Let $w_k = \left(M\left(q\left(\frac{\Delta^m x_k - L}{\lambda}\right)\right)\right)^{r_k}$ and $\mu_k = \frac{p_k}{r_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k < 1$ for all $k \in \mathbb{N}$. Let μ be such that $0 < \mu < \mu_k$ for all $k \in \mathbb{N}$. Define the sequences (u_k) and (v_k) as follows:

For $w_k \geq 1$, let $u_k = w_k$ and $v_k = 0$ and for $w_k < 1$, let $u_k = 0$ and $v_k = w_k$.

Then clearly for all $k \in \mathbb{N}$ and we have $w_k = u_k + v_k$, $w_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$,

$u_k^{\mu_k} \le u_k \le w_k$ and $v_k^{\mu_k} \le v_k^{\mu}$. Therefore

$$\frac{1}{n}\sum_{k=1}^{n} w_k^{\mu_k} < \frac{1}{n}\sum_{k=1}^{n} w_k + \left[\frac{1}{n}\sum_{k=1}^{n} v_k\right]^{\mu}$$

(for this inequality one may refer to Maddox [31]).

Hence $(x_k) \in W(\Delta^m, M, p, q)$.

This completes the proof of the theorem.

Theorem 2.11. The sequence spaces $W_0(M,p,q)$ and $W_\infty(M,p,q)$ are solid and monotone.

Remark 2.12. In general it is difficult to predict about the solidity of $W_0(\Delta^m, M, p, q)$ and $W_\infty(\Delta^m, M, p, q)$ when $m \geq 1$. For this consider the following example.

Example 2.13. Let $X = \mathbb{C}$, $M(x) = x$, $q(x) = |x|$, $m = 2$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then $x = (x_k) = (k) \in W_0(\Delta^2, M, p, q)$ but $\alpha x = (\alpha_k x_k) \notin W_0(\Delta^2, M, p, q)$ for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $W_0(\Delta^2, M, p, q)$ is not solid.

BIBLIOGRAPHY

[1] **Chen, S.T.,** Geometry of Orlicz spaces, ***Dissert. Math.,*** 356 (1996), 1-204.

[2] **Chen, S.T. and Sun, H.,** Extreme points and rotundity of sequence Orlicz spaces, ***Natur. Sci. J. Harbin Normal Univ.***, 1 (2) (1985), 1-6 (in Chinese).

[3] **Chen, S.T. and Wang Tingfu,** H-property of Orlicz spaces, ***Chinese Ann. Math.,*** 8A (1987), 61-67.

[4] **Clarkson, J.A.,** Uniformly convex spaces, ***Trans. Amer. Math. Soc.,*** 40 (1936), 396-414.

[5] **Connor, J.S.,** The statistical and strong p-Cesaro convergence of sequences, ***Analysis***, 8 (1988), 47-63.

[6] **Danes, J.,** A geometric theorem useful in non-linear functional analysis, ***Bull. Un. Math. Ital.,*** 6 (1972), 369-372.

[7] **Diestel, J.,** ***Geometry of Banach Spaces Selected Topics,*** Lecture Notes in Mathematics, Springer-Verlag, 485 (1975).

[8] **Et, M. and Colak, R.,** On some generalized difference sequence spaces, ***Soochow J. Math.,*** 21 (4) (1995), 377-386.

[9] **Fast, H.,** Surla convergence statistique, ***Colloq. Math.,*** 2 (1951), 241-244.

[10] **Freedman, A.R. and Sember, J.J.,** Densities and summability, ***Pacific J. Math.,*** 95 (1981), 293-305.

[11] **Fridy, J.A.,** On statistical convergence, ***Analysis,*** 5 (1985), 301-303.

[12] **Geobel, K. and Sekowski, T.,** ***The modulus of non-compact convexity,*** Ann. Univ. Maria Curie-Sklodowska, Seet., A 38 (1984), 41-48.

[13] **Grzaslewicz, R. and Hudzik, H.,** Smooth points of Orlicz spaces equipped with the Luxemburg norm, ***Math. Nachr.,*** 155 (1992), 31-45.

[14] **Hardy, G.H.,** ***Divergent Series,*** Oxford Univ. Press, 1949.

[15] **Hudzik, H.,** Orlicz spaces without strongly extreme points and without H-points, ***Canad. Math. Bull.,*** 36(2) (1993), 173-177.

[16] **Hudzik, H. and Denker, M.,** Uniformly non-$l'_{(n)}$ Musielak-Orlicz sequence spaces endowed with the Luxemburg norm, ***Proc. Indian Acad. Sci.,*** 2 (1991), 71-86.

[17] **Hudzik, H. and Kurc, W.,** Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices, ***J. Approx. Theory,*** 95 (1998), 353-368.

[18] **Hudzik, H. and Pallaschke, D.,** On some convexity properties of Orlicz sequence spaces equipped with the Luxemburg norm, ***Math. Nachr.,*** 186 (1997), 167-185.

[19] **Huff, R.,** Banach spaces which are nearly uniformly convex, ***Rockey Mountain J. Math.,*** 10 (1980), 473-749.

[20] **Kaminska, A.,** Uniform convexity of Musielak-Orlicz sequence spaces, ***J. Approx. Theory,*** 47 (1986), 302-322.

[21] **Kaminska, A.,** Rotundity of Musielak-Orlicz sequence spaces, ***Bull. Acad. Polon. Math.,*** 29 (1981), 137-144.

[22] **Khan, V.A.,** Some sequence spaces and their geometric properties, ***Ph.D. Thesis,*** (2004).

[23] **Kirk, W.A. and Sims, B.,** ***Handbook of Metric Fixed Point Theory,*** Kluwer Academic Publishers, (2001).

[24] **Kizmaz, H.,** On certain sequence spaces, ***Canad. Math. Bull.,*** 24 (1981), 169-176.

[25] **Kolk, E.,** The statistical convergence in Banach space, ***Acta et Comment. Univ. Tartu.,*** 928 (1991), 41-52.

[26] **Kolk, E.,** On strong boundedness and summability with respect to a sequence of moduli, ***Acta et Comment. Univ. Tartu.,*** 960 (1993), 41-50.

[27] **Kolk, E.,** Inclusion relations between the statistical convergence and strong summability, ***Acta et Comment. Univ. Tartu. Math.,*** 2 (1998), 39-54.

[28] **Kutzarowa, D.N.,** An isomorphic characterization of property (β) of Rolewicz, ***Note Math.,*** 10 (2) (1990), 347-354.

[29] **Kutzarowa, D.N. and Papini, P.L.,** On a characterization of property (β) and LUR, ***Bull. Un. Math. Ital.,*** A (7), No. 26 (1992), 209-214.

[30] **Lovaglia, A.R.,** Locally uniformly convex Banach spaces, ***Trans. Amer. Math. Soc.,*** 29 (1977), 963-970.

[31] **Maddox, I.J.,** Spaces of strongly summable sequences, ***Quart. J. Math. Oxford*** (2) 18 (1967), 161-166.

[32] **Maddox, I.J.,** ***Elements of Functional Analysis,*** Cambridge University Press, 1970.

[33] **Maddox, I.J.,** Sequence spaces defined by a modulus, ***Math. Proc. Camb. Phil. Soc.,*** 100 (1986), 161-166

[34] **Montesinos, V.**, Drop property equals reflexivity, ***Studia Math.***, 87 (1987), 93-100.

[35] **Musielak, J.**, ***Orlicz Spaces and Modular Spaces,*** Lecture Notes in Mathematics, 1034, (1983).

[36] **Panda, B.B. and Kapoor, O.P.**, Generalization of local uniform convexity of the norm, ***J. Math. Anal. Appl.,*** 52 (1975), 300-308.

[37] **Parashar, S.D. and Choudhary, B.**, Sequence spaces defined by Orlicz functions, ***Indian J. Pure Appl. Math.,*** 25 (4) (1994), 419-428.

[38] **Pluciennik, R., Wang, T. and Zhang, Y.**, H-points and denting points in Orlicz spaces, ***Comment. Math.,*** 33 (1993), 135-151.

[39] **Rolewicz, S.**, On drop property, ***Studia Math.,*** 85 (1987), 27-35.

[40] **Rolewicz, S.**, On Δ-uniform convexity and drop property, ***Studia Math.,*** 87 (1987), 181-191.

[41] **Ruckle, W.H.**, *FK* spaces in which the sequence of co-ordinate vectors is bounded, ***Canad. J. Math.,*** 25 (1973), 973-978.

[42] **Salat, T.**, On statistically convergent sequences of real numbers, ***Math. Slovaca,*** 30 (1980), 139-159.

[43] **Smith, M.A.**, Banach spaces that are uniformly rotund in weakly compact set of direction, ***Canad. J. Math.,*** 29 (1977), 963-970.

[44] **Wang Junming and Wang Tingfu,** *WM* property of Musielak-Orlicz sequence spaces, ***Far East J. Math. Sci.,*** 5 (3) (1997), 475-496.

[45] **Wu, C., Chen, S.T. and Wang, Y.**, H-property of sequence Orlicz spaces, ***J. Harbin Inst. Tech. (Math. Issue),*** (1985), 6-11 (in Chinese).